

8-th Hong Kong (China) Mathematical Olympiad 2005

December 3, 2005

1. A planet is inhabited by $3 \cdot 2005!$ aliens who speak 2005 distinct languages. Every two aliens speak exactly one language in common. Show that there are three aliens who speak the same language.
2. Let $4n$ segments of unit length be given inside a circle of radius n , where $n \in \mathbb{N}$. Let L be an arbitrary line. Show that there exists a line L' that is either parallel or perpendicular to L and which cuts at least two of the given segments.
3. Positive numbers a, b, c, d satisfy $a + b + c + d = 1$. Prove the inequality

$$6(a^3 + b^3 + c^3 + d^3) \geq (a^2 + b^2 + c^2 + d^2) + \frac{1}{8}.$$

4. Show that there exist infinitely many square-free positive integers n such that n divides $2005^n - 1$.