5-th Hong Kong (China) Mathematical Olympiad 2002

December 21, 2002

1. Two circles meet at points A and B. A line through B intersects the first circle again at K and the second circle at M. A line parallel to AM is tangent to the first circle at Q. The line AQ intersects the second circle again at R.

   (a) Prove that the tangent to the second circle at R is parallel to AK.

   (b) Prove that these two tangents meet on KM.

2. In a conference there are $n \geq 3$ mathematicians. Every two mathematicians communicate in one of the n official languages of the conference. For any three different official languages there exist three mathematicians who communicate with each other in these three languages. Find all n for which this is possible.

3. Let $a \geq b \geq c \geq 0$ be real numbers with $a+b+c=3$. Prove that
$$ab^2+bc^2+ca^2 \leq \frac{27}{8}$$
and find the cases of equality.

4. Let p be a prime number such that $p \equiv 1 \pmod 4$. Determine $\displaystyle\sum_{k=1}^{\frac{p-1}{2}} \left\{ \frac{k^2}{p} \right\}$, where $\{x\} = x - [x]$.

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