3-rd Hungary–Israel Binational Mathematical Competition 1992

Weizmann Institute, Rehovot, Israel

Individual competition – May 22

1. Prove that if c is a positive number distinct from 1 and n a positive integer, then
$$n^2 \leq \frac{c^n + c^{-n} - 2}{c + c^{-1} - 2}.$$

2. A set S consists of 1992 positive integers among whose units digits all 10 digits occur. Show that there is such a set S having no nonempty subset S_1 whose sum of elements is divisible by 2000.

3. We are given 100 strictly increasing sequences of positive integers: $A_i = (a_1^{(i)}, a_2^{(i)}, \dots)$, $i = 1, 2, \dots, 100$. For $1 \leq r, s \leq 100$ we define the following quantities:
$$\begin{aligned} f_r(u) &= \text{the number of elements of } A_r \text{ not exceeding } n; \\ f_{r,s}(u) &= \text{the number of elements of } A_r \cap A_s \text{ not exceeding } n. \end{aligned}$$
Suppose that $f_r(n) \geq \frac{1}{2}n$ for all r and n. Prove that there exists a pair of indices (r, s) with $r \neq s$ such that $f_{r,s}(n) \geq \frac{8n}{33}$ for at least five distinct n-s with $1 \leq n < 19920$.

4. A convex pentagon P with all vertices at lattice points is given on a coordinate plane. Let Q denote the convex pentagon bounded by its five diagonals. Show that there is a lattice point inside Q or on its boundary.

Team competition – May 25

We examine the following two sequences:

- The Fibonacci sequence: $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$;
- The Lucas sequence: $L_0 = 2$, $L_1 = 1$, $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$.

It is known that for all $n \geq 0$
$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}, \quad L_n = \alpha^n + \beta^n, \quad \text{where } \alpha = \frac{1+\sqrt{5}}{2}, \ \beta = \frac{1-\sqrt{5}}{2}.$$

These formulae can be used without proof.

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1. Prove that $1 + L_{2^j} \equiv 0 \pmod{2^{j+1}}$ for $j \geq 0$.

2. Prove that $\sum_{k=1}^{n} \left[\alpha^k F_k + \frac{1}{2}\right] = F_{2n+1}$ for $n > 1$.

3. We call a nonnegative integer r-Fibonacci number if it is a sum of r (not necessarily distinct) Fibonacci numbers. Show that there infinitely many positive integers that are not r-Fibonacci numbers for any r, $1 \leq r \leq 5$.

4. Prove that $F_{n-1}F_nF_{n+1}L_{n-1}L_nL_{n+1}$ $(n \geq 2)$ is not a perfect square.

5. Show that $L_{2n+1} + (-1)^{n+1}$ $(n \geq 1)$ can be written as a product of three (not necessarily distinct) Fibonacci numbers.

6. The coordinates of all vertices of a given rectangle are Fibonacci numbers. Suppose that the rectangle is not such that one of its vertices is on the x-axis and another on the y-axis. Prove that either the sides of th rectangles are parallel to the axes, or make an angle of $45°$ with the axes.

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