12-th Hungary–Israel Binational Mathematical Competition 2001

First Day

1. Find positive integers x, y, z such that $x > z > 1999 \cdot 2000 \cdot 2001 > y$ and $2000x^2 + y^2 = 2001z^2$.

2. Points A, B, C, D lie on a line l, in that order. Find the locus of points P in the plane for which $\angle APB = \angle CPD$.

3. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$,
$$f(f(x)) = f(x) + x.$$

Second Day

4. Let $P(x) = x^3 - 3x + 1$. Find the polynomial Q whose roots are the fifth powers of the roots of P.

5. In a triangle ABC, B_1 and C_1 are the midpoints of AC and AB respectively, and I is the incenter. The lines B_1I and C_1I meet AB and AC respectively at C_2 and B_2. If the areas of $\triangle ABC$ and $\triangle AB_2C_2$ are equal, find $\angle BAC$.

6. Let be given 32 positive integers with the sum 120, none of which is greater than 60. Prove that these integers can be divided into two disjoint subsets with the same sum of elements.

Team competition

Here G_n denotes a simple undirected graph with n vertices, K_n denotes the complete graph with n vertices, $K_{n,m}$ the complete bipartite graph whose components have m and n vertices, and C_n a circuit with n vertices. The number of edges in the graph G_n is denoted $e(G_n)$.

1. The edges of K_n ($n \geq 3$) are colored with n colors, and every color is used. Show that there is a triangle whose sides have different colors.

2. If $n \geq 5$ and $e(G_n) \geq \dfrac{n^2}{4} + 2$, prove that G_n contains two triangles that share exactly one vertex.

3. If $e(G_n) \geq \dfrac{n\sqrt{n}}{2} + \dfrac{n}{4}$, prove that G_n contains C_4.

4. (a) If G_n does not contain $K_{2,3}$, prove that $e(G_n) \leq \dfrac{n\sqrt{n}}{\sqrt{2}} + n$.

(b) Given $n \geq 16$ distinct points $P_1, \dots, P_n$ in the plane, prove that at most $n\sqrt{n}$ of the segments P_iP_j have unit length.

5. (a) Let p be a prime. Consider the graph whose vertices are the ordered p[airs (x,y) with $x,y \in \{0,1,\dots,p-1\}$ and whose edges join vertices (x,y) and (x',y') if and only if $xx'+yy' \equiv 1 \pmod{p}$. Prove that this graph does not contain C_4.

   (b) Prove that for infinitely many values n there is a graph G_n with $e(G_n) \geq \frac{n\sqrt{n}}{2} - n$ that does not contain C_4.