11-th Hungary–Israel Binational Mathematical Competition 2000

First Day

1. Let S be the set of all partitions of 2000 (in a sum of positive integers). For every such partition p, we define $f(p)$ to be the sum of the number of summands in p and the maximal summand in p. Compute the minimum of $f(p)$ when $p \in S$.

2. Prove or disprove: For any positive integer k there exists an integer $n > 1$ such that the binomial coefficient $\binom{n}{i}$ is divisible by k for any $1 \leq i \leq n-1$.

3. Let ABC be a non-equilateral triangle. The incircle is tangent to the sides BC, CA, AB at A_1, B_1, C_1, respectively, and M is the orthocenter of triangle $A_1B_1C_1$. Prove that M lies on the line through the incenter and circumcenter of $\triangle ABC$.

Second Day

4. Let A and B be two subsets of $S = \{1, 2, \dots, 2000\}$ with $|A| \cdot |B| \geq 3999$. For a set X, let $X - X$ denotes the set $\{s - t \mid s, t \in X,\ s \neq t\}$. Prove that $(A - A) \cap (B - B)$ is nonempty.

5. For a given integer d, let us define $S = \{m^2 + dn^2 \mid m, n \in \mathbb{Z}\}$. Suppose that p, q are two elements of S, where p is prime and $p \mid q$. Prove that $r = q/p$ also belongs to S.

6. Let k and l be two given positive integers and a_{ij} ($1 \leq i \leq k$, $1 \leq j \leq l$) be kl positive integers. Show that if $q \geq p > 0$, then
$$\left(\sum_{j=1}^{l}\left(\sum_{i=1}^{k} a_{ij}^{p}\right)^{q/p}\right)^{1/q} \leq \left(\sum_{i=1}^{k}\left(\sum_{j=1}^{l} a_{ij}^{q}\right)^{p/q}\right)^{1/p}.$$

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com