Greek Team Selection Test 2005

Athens, April 2, 2005

Juniors

1. Is it possible to cut a convex 39-gon into nine convex hexagons?
2. Prove that if x, y, z are real numbers, then
$$\frac{x^2-y^2}{2x^2+1}+\frac{y^2-z^2}{2y^2+1}+\frac{z^2-x^2}{2z^2+1}\leq 0.$$
3. In a cyclic quadrilateral $ABCD$, M is the midpoint of side AB and R is the intersection point of MC and BD. A line through C is parallel to AR and meets BD at point S. Prove that if $\angle CAD = \angle RAB = \frac{1}{2}\angle BMC$, then $BR = SD$.
4. Find all positive integers $n > 3$ for which n divides $(n-2)!$.

Seniors

1. The sides of a triangle are the roots of a cubic equation with rational coefficients. Prove that the altitudes of this triangle are roots of a 6-th degree equation with rational coefficients.
2. The circle Γ and the line e have no common points. Let AB be the diameter of Γ perpendicular to e, with B closer to e than A. An arbitrary point $C \neq A, B$ is chosen on Γ. The line AC intersects e at D. The line DE is tangent to Γ at E, with B and E on the same side of AC. Let BE intersect e at F, and let AF intersect Γ again at $G \neq A$. Prove that the reflection of G in AB lies on the line CF.
3. Consider the polynomial $P(x) = x^3 + 19x^2 + 94x + a$, where a is a positive integer. If p is a prime number, prove that among the numbers $P(0), P(1), \dots, P(p-1)$ at most three are divisible by p.
4. There are 10001 students at a university. Some students join together to form several clubs (a student may belong to different clubs). Some clubs join together to form several societies (a club may belong to different societies). There are a total of k societies. Suppose that the following conditions hold:

   (i) Each pair of students are in exactly one club.

   (ii) For each student and each society, the student is in exactly one club of the society.

   (iii) Each club has an odd number of students. In addition, a club with $2m+1$ students (m is a positive integer) is in exactly m societies.

   Find all possible values of k.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com