20-th Hellenic Mathematical Olympiad 2003

February 15, 2003

Juniors

1. Find all positive integers n for which number $A = n^3 - n^2 + n - 1$ is prime.

2. Find all four-digit natural numbers $\overline{xyzw}$ with the property that their sum with the sum of their digits equals 2003.

3. In an isosceles triangle ABC with $AB = AC$, AH is the altitude and M the circumcenter. The line through M parallel to AB meets BC at D. The circumcircle of triangle AMD intersects the perpendicular bisector of AB again at S. Prove that $BS \parallel AM$ and that $AMBS$ is a rhombus.

4. Find all positive integers which can be written in the form $\dfrac{mn+1}{m+n}$, where m,n are positive integers.

Seniors

1. If a,b,c,d are positive numbers satisfying $a^3 + b^3 + ab = c + d = 1$, prove that

$$\left(a+\frac{1}{a}\right)^3 + \left(b+\frac{1}{b}\right)^3 + \left(c+\frac{1}{c}\right)^3 + \left(d+\frac{1}{d}\right)^3 \geq 40.$$

2. Find all real solutions of the system

$$\begin{aligned} x^2 + y^2 - z(x+y) &= 2, \\ y^2 + z^2 - x(y+z) &= 4, \\ z^2 + x^2 - y(z+x) &= 8. \end{aligned}$$

3. Given are a circle $\mathscr{C}$ with center K and radius r, point A on the circle and point R in its exterior. Consider a variable line e through R that intersects the circle at two points B and C. Let H be the orthocenter of triangle ABC. Show that there is a unique point T in the plane of circle $\mathscr{C}$ such that the sum $HA^2 + HT^2$ remains constant (as e varies.)

4. On the set Σ of points of the plane Π we define the operation $*$ which maps each pair (X,Y) of points in Σ to the point $Z = X * Y$ that is symmetric to X with respect to Y. Consider a square $ABCD$ in Π. Is it possible, using the points A,B,C and applying the operation $*$ finitely many times, to construct the point D?

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