18-th Hellenic Mathematical Olympiad 2001

Athens, February 10, 2001

Juniors

1. Let α, β, x, y be real numbers such that $\alpha + \beta = 1$. Prove that

$$\frac{1}{\alpha/x + \beta/y} \leq \alpha x + \beta y$$

and find when equality holds.

2. (a) Find all pairs (m, n) of integers satisfying $m^3 - 4mn^2 = 8n^3 - 2m^2 n$.

   (b) Among such pairs find those for which $m + n^2 = 3$.

3. We are given 8 different weights and a balance without a scale.

   (a) Find the smallest number of weighings necessary to find the heaviest weight.

   (b) How many weighting is further necessary to find the second heaviest weight?

4. Let $A\Delta$ be an altitude of a triangle $AB\Gamma$. The bisectors AE, BZ of angles at A and B ($E \in B\Gamma$, $Z \in A\Gamma$) meet at I. Let Θ be the foot of perpendicular from I to $A\Gamma$. Also, let ξ be the line through A perpendicular to $A\Gamma$. If the line $E\Theta$ intersects ξ at K, prove that $A\Delta = AK$.

Seniors

1. A triangle $AB\Gamma$ is inscribed in a circle of radius R. Let $B\Delta$ and ΓE be the bisectors of the angles B at Γ respectively and let the line ΔE meet the arc AB not containing Γ at point K. Let A_1, B_1, Γ_1 be the feet of perpendiculars from K to $B\Gamma$, $A\Gamma$, AB, and x, y are the distances from Δ and E to AB, respectively, then:

   (a) Express the lengths of $KA_1, KB_1, K\Gamma_1$ in terms of x, y and the ratio $\lambda = K\Delta/E\Delta$.

   (b) Prove that $\dfrac{1}{KB} = \dfrac{1}{KA} + \dfrac{1}{K\Gamma}$.

2. Prove that there are no positive integers α, β such that $(15\alpha + \beta)(\alpha + 15\beta)$ is a power of 3.

3. A function $f : \mathbb{N}_0 \to \mathbb{R}$ satisfies $f(1) = 3$ and

$$f(m+n) + f(m-n) - m + n - 1 = \frac{f(2m) + f(2n)}{2}$$

for any nonnegative integers m, n with $m \geq n$. Find all such functions f.

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4. The numbers 1 to 500 are written on a board. Two pupils A and B play the following game. A player in turn deletes one of the numbers from the board. The game is over when only two numbers remain. Player B wins if the sum of the two remaining numbers is divisible by 3, otherwise A wins. If A plays first, show that B has a winning strategy.