39-th German Mathematical Olympiad 2000

4-th Round – Berlin, May 7–10

Grade 10

First Day

1. Let M be the set of numbers that can be represented as the product of five consecutive even numbers.
   (a) Find the greatest integer t that divides every number in M.
   (b) Find the greatest natural number m such that every number in M has at least m positive divisors.
   (c) Find the smallest number in M having more than 200 positive divisors.
2. Let s be the sum of the reciprocals of the squares of the numbers from 1 to 2000. Prove that $s < 2$.
3. A cube $ABCDEFGH$ ($AE \parallel BF \parallel CG \parallel DH$) is divided into parts by planes AFH, BEG, CFH, DEG, EBD, FAC, GBD, HAC.
   (a) Find the number of these parts.
   (b) Compute the volume of each of the parts.

Second Day

4. In a right-angled triangle ABC, the incircle is tangent to the hypothenuse AB at P. Denote $d = AP$ and $e = BP$. Prove that the area F of the triangle is equal to $F = de$.
5. Let $P(x) = a_{20}x^{20} + \cdots + a_1x + a_0$ be a polynomial of degree 20 with integer coefficients, and let p,q be prime numbers such that there are no other prime numbers between p and q. It is given that $P(p) = 1234$ and $P(q) = 4321$. Decide if this uniquely determines the numbers p and q, and find them.
6. For decoration of a room, 100 different bands are needed to be hanged vertically. We dispose of four types of small pieces: red and blue ones of lengths 10cm and 20cm. These pieces are to be concatenated to form bands, but the pieces may not be of diferent colors. How long should the bands be a least, in order to be possible to make 100 different bands?

Grades 11-13

First Day

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1. For each real parameter a, find the number of real solutions to the system

$$\begin{aligned} |x|+|y| &= 1, \\ x^2+y^2 &= a. \end{aligned}$$

2. For an integer $n \geq 2$, find all real numbers x for which the polynomial

$$f(x)=(x-1)^4+(x-2)^4+\cdots+(x-n)^4$$

takes its minimum value.

3. Suppose that an interior point O of a triangle ABC is such that the angles $\angle BAO$, $\angle CBO$, $\angle ACO$ are all greater than or equal to 30°. Prove that the triangle ABC is equilateral.

Second Day

4. Find all nonnegative solutions (x,y,z) to the system

$$\begin{aligned} \sqrt{x+y}+\sqrt{z}=7, \\ \sqrt{x+z}+\sqrt{y}=7, \\ \sqrt{y+z}+\sqrt{x}=5. \end{aligned}$$

5. (a) Let be given $2n$ distinct points on a circumference, n of which are red and n are blue. Prove that one can join these points pairwise by n segments so that no two segments intersect and the endpoints of each segments have different colors.

   (b) Show that the statement from (a) remains valid if the points are in an arbitrary position in the plane so that no three of them are collinear.

6. A sequence (a_n) satisfies the following conditions:

   (i) For each $m \in \mathbb{N}$ it holds that $a_{2^m} = 1/m$.

   (ii) For each natural $n \geq 2$ it holds that $a_{2n-1}a_{2n} = a_n$.

   (iii) For all integers m,n with $2^m > n \geq 1$ it holds that $a_{2n}a_{2n+1} = a_{2^m+n}$.

   Determine a_{2000}. You may assume that such a sequence exists.

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