26-th German Federal Mathematical Competition 1995/96

Second Round

1. For a given set of points in space it is allowed to mirror a point from the set with respect to another point from the set, and to include the image in the set. Starting with a set of seven vertices of a cube, is it possible to include the eight vertex in the set after finitely many such steps?

2. The sequence $z_0, z_1, z_2, \dots$ is defined by $z_0 = 0$ and

$$z_n = \begin{cases} z_{n-1} + \dfrac{3^r - 1}{2} & \text{if } n = 3^{r-1}(3k+1) \text{ for some integers } r, k; \\ z_{n-1} - \dfrac{3^r + 1}{2} & \text{if } n = 3^{r-1}(3k+2) \text{ for some integers } r, k. \end{cases}$$

   Prove that every integer occurs exactly once in this sequence.

3. Rectangles ABB_1A_1, BCC_1A_2, CAA_2C_2 are constructed in the exterior of a triangle ABC. Prove that the perpendicular bisectors of the segments A_1A_2, B_1B_2, and C_1C_2 are concurrent.

4. Let p be an odd prime number. Find the positive integers x and y with $x \leq y$ for which $\sqrt{2p} - \sqrt{x} - \sqrt{y}$ is the smallest possible.

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