4-th German Federal Mathematical Competition 1973/74

Second Round

1. On a plane are given 25 points such that among any three of them, some two are on a distance smaller than 1. Show that among the given points there are 13 that can be covered by a disk of radius 1. Also prove the generalization of this statement.

2. There are 30 apparently equal balls, 15 of which have the weight a and the remaining 15 have the weight b, $a \neq b$. The balls are to be partitioned into two groups of 15, according to the weight. An assistant partitioned them into two groups, and we wish to check if his partition is correct. How can we check that with as few weighings as possible?

3. A circle K_1 of radius $1/2$ is inscribed in a semi-circle H with diameter AB and radius 1. A sequence of different circles $K_1, K_2, \dots$ with radii $r_1, r_2, \dots$ respectively are drawn so that, for each $n \geq 1$, circle K_{n+1} is tangent to H, K_n, and AB. Prove that $a_n = 1/r_n$ is an integer for each $n \in \mathbb{N}$, and that it is a perfect square for n even and two times a perfect square for n odd.

4. Peter and Paul gamble as follows. For each natural number, successively, they determine its largest odd divisor and compute its remainder when divided by 4. If this remainder is 1, then Peter pays Paul 1 DM (Deutsche mark); otherwise Paul pays Peter 1 DM. After some time they stop playing and balance the accounts. Prove that Paul wins.

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