33-rd German Federal Mathematical Competition 2002/03

Second Round

1. Suppose that the graph of a function $f : \mathbb{R} \to \mathbb{R}$ has two or more centers of symmetry. Prove that f is the sum of a linear and a periodic function.

2. The sequence $(a_n)_{n=1}^{\infty}$ is defined by

$$a_1 = 1,\ a_2 = 1,\ a_3 = 2, \quad \text{and} \quad a_{n+3} = \frac{a_{n+1}a_{n+2} + 7}{a_n} \text{ for } n > 0.$$

   Prove that all terms of this sequence are integers.

3. The diagonals of a convex cyclic quadrilateral $ABCD$ intersect at S. Let E and F be the projections of S on the sides AB and CD, respectively. Show that the perpendicular bisector of EF bisects sides BC and DA.

4. Let p and q be two different coprime positive integers. The set of integers is to be partitioned into three subsets A, B, C such that for every integer z, the numbers z, $z + p$ and $z + q$ belong to three different subsets. Show that such a partition is possible if and only if $p + q$ is divisible by 3.

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