28-th German Federal Mathematical Competition 1997/98

First Round

1. In the playboard shown beside, players A and B alternately fill the empty cells by integers, player A starting. In each step the empty cell and the integer can be chosen arbitrarily. Show that player A can always achieve that all the equalities hold after the last step.

$$
\begin{gathered}
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\end{gathered}
$$

2. Prove that there exists an infinite sequence of perfect squares with the following properties:

   (i) The arithmetic mean of any two consecutive terms is a perfect square;

   (ii) Every two consecutive terms are coprime;

   (iii) The sequence is strictly increasing.

3. Two squares are constructed outwardly on the sides BC and CA of a triangle ABC. Let M be the midpoint of AB and P and Q be the centers of the two squares. Prove that MPQ is an isosceles right triangle.

4. Prove that $n+\left[(\sqrt{2}+1)^n\right]$ is an odd number for every positive integer n.

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