31-st German Federal Mathematical Competition 2000/01

First Round

1. A heap of 2001 checkers is lying on a table. The heap must be transformed into heaps of three checkers step by step. In every step one can select one heap, remove one checker from it and split the remaining heap into two heaps. Could the target be achieved by a sequence of (completely performed) steps?

2. A sequence of real numbers $a_0, a_1, a_2, \ldots$ satisfies

$$a_0 = 1 \quad \text{and} \quad a_{n+1} = a_n + \sqrt{a_{n+1} + a_n} \quad \text{for all } n \in \mathbb{N}.$$

Prove that the sequence with this property is unique and find an explicit formula for a_n.

3. Let ABC be an acute-angled triangle with circumcenter O. Line BO intersects the circumcircle of $\triangle ABC$ again in D, and the altitude through A intersects the circumcircle in E. Show that quadrilateral $BECD$ and triangle ABC have the same area.

4. Prove that for every positive integer, the number of divisors whose last decimal digit is 1 or 9 is not smaller than the number of divisors whose last digit is 3 or 7.