French IMO Selection Test 2007

First Day

1. For a positive integer a, denote by a' the integer obtained by the following method: the decimal writing of a' is the inverse of the decimal writing of a (the decimal writing of a' can begin by zeros, unlike the one of a).

   Given a positive integer a_1, denote by a_n the sequence defined by $a_{n+1} = a_n + a'_n$.

   Can a_7 be prime?

2. If $a, b, c, d \in \mathbb{R}_+$ satisfy $a + b + c + d = 1$ prove that

$$6(a^3 + b^3 + c^3 + d^3) \geq a^2 + b^2 + c^2 + d^2 + \frac{1}{8}.$$

3. No two sides of a cyclic quadrilateral $ABCD$ are parallel. Let $E = AC \cap BD$ and $F = AD \cap BC$. Prove that C, D, E, F lie on a circle if and only if $EF \perp AB$.

Second Day

4. Is it possible to choose 5 points in the space in such a way that for each integer n, $1 \leq n \leq 10$, there are two among the chosen points whose distance is exactly n?

5. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that:

$$f(x - y + f(y)) = f(x) + f(y), \quad \text{for all } x, y \in \mathbb{Z}.$$

6. Angles of $\triangle ABC$ satisfy $\angle C < \angle A < 90^\circ$. Let D be a point on AC such that $BD = BA$ and denote by K and L the points of tangency of the incircle of $\triangle ABC$ with the sides AB and AC respectively. If J is the incenter of the incircle of $\triangle BCD$ prove that KL bisects AJ.

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