

French IMO Selection Test 2005

First Day

1. Suppose that positive integers x and y satisfy $3x^2 + x = 4y^2 + y$. Prove that $x - y$ is a perfect square.
2. Two rectangular triangles of respective areas S and S' are given such that the incircle of the first is congruent to the circumcircle of the second. Prove that

$$\frac{S}{S'} \geq 3 + 2\sqrt{2}.$$

3. At an international conference with $n \geq 3$ participants, 14 languages are spoken. It is known that:

- (i) every three participants speak one language in common;
- (ii) each language is spoken by at most half the participants.

What is the smallest possible value of n ?

Second Day

4. Let X be a nonempty set of natural numbers. Suppose that for all $x \in X$

$$4x \in X \quad \text{and} \quad \lfloor \sqrt{x} \rfloor \in X.$$

Show that $X = \mathbb{N}$.

5. In a triangle ABC with $BC = AC + \frac{1}{2}AB$, P is a point on side AB such that $AP = 3PB$. Prove that $\angle PAC = 2\angle CPA$.
6. Let $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$ be a polynomial of degree $n \geq 5$ with integer coefficients. Suppose that P has n distinct integer roots $0, \alpha_2, \dots, \alpha_n$. Find all integers k such that $P(P(k)) = 0$.