French IMO Selection Test 2002

First Day

1. In an acute-angled triangle ABC, A_1 and B_1 are the feet of the altitudes from A and B respectively, and M is the midpoint of AB.
   (a) Prove that MA_1 is tangent to the circumcircle of triangle A_1B_1C.
   (b) Prove that the circumcircles of triangles $A_1B_1C_1$, BMA_1, and AMB_1 have a common point.

2. Consider the set S of integers k which are products of four distinct primes. Such an integer $k = p_1p_2p_3p_4$ has 16 positive divisors $1 = d_1 < d_2 < \cdots < d_{15} < d_{16} = n$. Find all elements of S less than 2002 such that $d_9 - d_8 = 22$.

3. Let n be a positive integer and let $(a_1, a_2, \dots, a_{2n})$ be a permutation of $1, 2, \dots, 2n$ such that the numbers $|a_{i+1} - a_i|$ are pairwise distinct for $i = 1, \dots, 2n-1$.
   (a) Prove that if $a_2, a_4, \dots, a_{2n}$ are in the set $\{1, \dots, n\}$, then $a_1 - a_{2n} = n$.
   (b) Prove the converse.

Second Day

4. There are three countries and n mathematicians in each of them. Suppose that every mathematician knows at least $n+1$ mathematicians from the other two countries. Prove that there exist three mathematicians who know each other.

5. In a non-equilateral triangle ABC, I is the incenter and O the circumcenter. Prove that $\angle AIO \le 90^\circ$ if and only if $2BC \le AB + AC$.

6. Let $p \ge 3$ be a prime number. Show that there exist p positive integers $a_1, a_2, \dots, a_p$ not exceeding $2p^2$ such that the $\frac{p(p-1)}{2}$ sums $a_i + a_j$ $(i < j)$ are all distinct.

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