French Mathematical Olympiad 1999

Time: 5 hours.

1. What is the maximum possible volume of a cylinder inscribed in a cone and having the same axis of symmetry as the cone? What is the maximum possible volume of a ball inscribed in the cone with center on the axis of symmetry of the cone? Compare these three volumes.

2. Find all natural numbers n such that

$$(n+3)^n = \sum_{k=3}^{n+2} k^n.$$

3. For which acute-angled triangles is the ratio of the smallest side to the inradius the maximum?

4. On a table are given 1999 red candies and 6661 yellow candies. The candies are indistinguishable due to the same packing. A gourmet applies the following procedure as long as it is possible:

   (i) He picks any of the remaining candies, notes its color, eats it and goes to (ii).

   (ii) He picks any of the remaining candies, and notes its color: if it is the same as the color of the last eaten candy, eats it and goes to (ii); otherwise returns it upon repacking and goes to (i).

   Prove that all the candies will be eaten and find the probability that the last eaten candy will be red.

5. Prove that the points symmetric to the vertices of a triangle with respect to the opposite side are collinear if and only if the distance from the orthocenter to the circumcenter is twice the circumradius.

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