35-th Czech and Slovak Mathematical Olympiad 1986

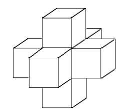
1. Given $n \in \mathbb{N}$, let $\mathscr{A}$ be a family of subsets of $\{1,2,\dots,n\}$. If for every two sets $B,C \in \mathscr{A}$ the set $(B\cup C)\setminus(B\cap C)$ has an even number of elements, find the largest possible number of elements of $\mathscr{A}$.

2. Let $P(x)$ be a polynomial with integer coefficients of degree $n \geq 3$. If $x_1,\dots,x_m$ ($m \geq 3$) are different integers such that

$$P(x_1) = P(x_2) = \cdots = P(x_m) = 1,$$

prove that P cannot have integer roots.

3. Prove that the entire space can be partitioned into "crosses" made of seven unit cubes as shown in the picture.



4. Let C_1, C_2, and C_3 be points inside a bounded convex planar set $\mathscr{M}$. Rays l_1, l_2, l_3 emanating from C_1, C_2, C_3 respectively partition the complement of the set $\mathscr{M} \cup l_1 \cup l_2 \cup l_3$ into three regions $\mathscr{D}_1, \mathscr{D}_2, \mathscr{D}_3$. Prove that if the convex sets A and B satisfy

$$A\cap l_j = \emptyset = B \cap l_j \text{ and } A\cap \mathscr{D}_j \neq \emptyset \neq B\cap \mathscr{D}_j \quad \text{for } j = 1,2,3,$$

then $A\cap B \neq \emptyset$.

5. A sequence of natural numbers $a_1, a_2, \dots$ satisfies $a_1 = 1$, $a_{n+2} = 2a_{n+1} - a_n + 2$ for $n \in \mathbb{N}$. Prove that for every natural n there exists a natural m such that $a_n a_{n+1} = a_m$.

6. Assume that $M \subset \mathbb{N}$ has the property that every two numbers m,n of M satisfy $|m-n| \geq mn/25$. Prove that the set M contains no more than 9 elements.

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