54-th Czech and Slovak Mathematical Olympiad 2005

Third Round – Benešov, April 3–6, 2005

Category A

1. Consider all arithmetical sequences of real numbers $(x_i)_{i=1}^{\infty}$ and $(y_i)_{i=1}^{\infty}$ with the common first term, such that for some $k > 1$,

$$x_{k-1}y_{k-1} = 42, \quad x_k y_k = 30, \quad \text{and} \quad x_{k+1}y_{k+1} = 16.$$

Find all such pairs of sequences with the maximum possible k. *(J. Šimša)*

2. Determine for which m there exist exactly 2^{15} subsets X of $\{1, 2, \dots, 47\}$ with the following property: m is the smallest element of X, and for every $x \in X$, either $x + m \in X$ or $x + m > 47$. *(R. Kučera)*

3. In a trapezoid $ABCD$ with $AB \parallel CD$, E is the midpoint of BC. Prove that if the quadrilaterals $ABED$ and $AECD$ are tangent, then the sides $a = AB$, $b = BC$, $c = CD$, $d = DA$ of the trapezoid satisfy the equalities

$$a + c = \frac{b}{3} + d \quad \text{and} \quad \frac{1}{a} + \frac{1}{c} = \frac{3}{b}.$$

(R. Horenský)

4. An acute-angled triangle AKL is given on a plane. Consider all rectangles $ABCD$ circumscribed to triangle AKL such that point K lies on side BC and point L lies on side CD. Find the locus of the intersection S of the diagonals AC and BD. *(J. Šimša)*

5. Let p, q, r, s be real numbers with $q \neq -1$ and $s \neq -1$. Prove that the quadratic equations $x^2 + px + q = 0$ and $x^2 + rx + s = 0$ have a common root, while their other roots are inverse of each other, if and only if

$$pr = (q+1)(s+1) \quad \text{and} \quad p(q+1)s = r(s+1)q.$$

(A double root is counted twice.) *(J. Švrček)*

6. Decide whether for every arrangement of the numbers $1, 2, 3, \dots, 15$ in a sequence one can color these numbers with at most four different colors in such a way that the numbers of each color form a monotone subsequence. *(J. Šimša)*

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