# 8-th Croatian National Mathematical Competition 1999

## High School

Supetar, May 5–8, 1999

### 1-st Grade

1. Circles $k_1$ and $k_2$ with radii $r_1 = 6$ and $r_2 = 3$ are externally tangent and touch a circle $k$ with radius $r = 9$ from inside. A common external tangent of $k_1$ and $k_2$ intersects $k$ at $P$ and $Q$. Determine the length of $PQ$.

2. If $a, b, c$ are positive numbers with $a + b + c = 1$, prove the inequality
$$\frac{a^3}{a^2+b^2} + \frac{b^3}{b^2+c^2} + \frac{c^3}{c^2+a^2} \geq \frac{1}{2}\,.$$

3. For each $a$, $1 < a < 2$, the graphs of functions $y = 1 - |x-1|$ and $y = |2x-a|$ determine a figure. Prove that the area of this figure is less than $1/3$.

4. A triple of numbers $(a_1, a_2, a_3) = (3, 4, 12)$ is given. The following operation is performed a finite number of times: choose two numbers $a, b$ from the triple and replace them by $0.6x - 0.8y$ and $0.8x + 0.6y$. Is it possible to obtain the (unordered) triple $(2, 8, 10)$?

### 2-nd Grade

1. In a triangle $ABC$, the inner and outer angle bisectors at $C$ intersect the line $AB$ at $L$ and $M$, respectively. Prove that if $CL = CM$ then $AC^2 + BC^2 = 4R^2$, where $R$ is the circumradius of $\triangle ABC$.

2. For a real parameter $a$, solve the equation $x^4 - 2ax^2 + x + a^2 - a = 0$. Find all $a$ for which all solutions are real.

3. Let $a, b, c$ be positive real numbers with $abc = 1$. Prove the inequality
$$a^{b+c} b^{c+a} c^{a+b} \leq 1.$$

4. On a basketball competition $n$ teams took part. Each pair of teams played exactly one match, and there were no draws. At the end of the competition the $i$-th team had $x_i$ wins and $y_i$ defeats ($i = 1, \dots, n$). Prove that $x_1^2 + x_2^2 + \cdots + x_n^2 = y_1^2 + y_2^2 + \cdots + y_n^2$.



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## 3-rd Grade

1. A triangle $ABC$ is inscribed in a rectangle $APQR$ so that points $B$ and $C$ lie on segments $PQ$ and $QR$, respectively. If $\alpha, \beta, \gamma$ are the angles of the triangle, prove that
$$\cot\alpha \cdot S_{BCQ} = \cot\beta \cdot S_{ACR} + \cot\gamma \cdot S_{ABP}.$$

2. The base of a pyramid $ABCDV$ is a rectangle $ABCD$ with the sides $AB = a$ and $BC = b$, and all lateral edges of the pyramid have length $c$. Find the area of the intersection of the pyramid with a plane that contains the diagonal $BD$ and is parallel to $VA$.

3. The vertices of a triangle with sides $a \geq b \geq c$ are centers of three circles, such that no two of the circles have common interior points and none contains any other vertex of the triangle. Determine the maximum possible total area of these three circles.

4. Given nine positive integers, is it always possible to choose four different numbers $a, b, c, d$ such that $a + b$ and $c + d$ are congruent modulo 20?

## 4-th Grade

1. For every edge of a tetrahedron, we consider a plane through its midpoint that is perpendicular to the opposite edge. Prove that these six planes intersect in a point symmetric to the circumcenter of the tetrahedron with respect to its centroid.

2. Let $n > 1$ be an integer. Find the number of permutations $(a_1, a_2, \dots, a_n)$ of the numbers $1, 2, \dots, n$ such that $a_i > a_{i+1}$ holds for exactly one $i \in \{1, 2, \dots, n-1\}$.

3. Let $(a_n)$ be defined by $a_1 = a_2 = 1$ and $a_n = a_{n-1} + a_{n-2}$ for $n > 2$. Compute the sum $\frac{a_1}{2} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + \cdots$.

4. On the coordinate plane is given the square with vertices $T_1(1,0)$, $T_2(0,1)$, $T_3(-1,0)$, $T_4(0,-1)$. For every $n \in \mathbb{N}$, point $T_{n+4}$ is defined as the midpoint of the segment $T_n T_{n+1}$. Determine the coordinates of the limit point of $T_n$ as $n \to \infty$, if it exists.



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