14-th Croatian National Mathematical Competition 2005

High School

Omišalj on Krk, May 4–7, 2005

1-st Grade

1. Find all possible digits x, y, z such that the number $\overline{13xy45z}$ is divisible by 792.
2. The lines joining the incenter of a triangle to the vertices divide the triangle into three triangles. If one of these triangles is similar to the initial one, determine the angles of the triangle.
3. If k, l, m are positive integers with $\frac{1}{k} + \frac{1}{l} + \frac{1}{m} < 1$, find the maximum possible value of $\frac{1}{k} + \frac{1}{l} + \frac{1}{m}$.
4. The circumradius R of a triangle with side lengths a, b, c satisfies $R = \dfrac{a\sqrt{bc}}{b+c}$. Find the angles of the triangle.

2-nd Grade

1. Let $a \neq 0$, b, c be real numbers. If x_1 is a root of the equation $ax^2 + bx + c = 0$ and x_2 a root of $-ax^2 + bx + c = 0$, show that there is a root x_3 of $\dfrac{a}{2}x^2 + bx + c = 0$ between x_1 and x_2.
2. Let U be the incenter of a triangle ABC and O_1, O_2, O_3 be the circumcenters of the triangles BCU, CAU, ABU, respectively. Prove that the circumcircles of the triangles ABC and $O_1O_2O_3$ have the same center.
3. If a, b, c are real numbers greater than 1, prove that for any real number r
$$(\log_a bc)^r + (\log_b ca)^r + (\log_c ab)^r \geq 3 \cdot 2^r.$$
4. Show that in any set of eleven integers there are six whose sum is divisible by 6.

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3-rd Grade

1. Find all positive integer solutions of the equation $k!l! = k! + l! + m!$.

2. The incircle of a triangle ABC touches AC, BC, and AB at M, N, and R, respectively. Let S be a point on the smaller arc MN and t be the tangent to this arc at S. The line t meets NC at P and MC at Q. Prove that the lines AP, BQ, SR, MN have a common point.

3. Find the locus of points inside a trihedral angle such that the sum of their distances from the faces of the trihedral angle has a fixed positive value a.

4. The vertices of a regular 2005-gon are colored red, white and blue. Whenever two vertices of different colors stand next to each other, we are allowed to recolor them into the third color.

   (a) Prove that there is a finite sequence of allowed recolorings after which all the vertices are of the same color.

   (b) Is that color uniquely determined by the initial coloring?

4-th Grade

1. A sequence (a_n) is defined by $a_1 = 1$ and $a_n = a_1 a_2 \cdots a_{n-1} + 1$ for $n \geq 2$. Find the smallest real number M such that

$$\sum_{n=1}^{m} \frac{1}{a_n} < M \quad \text{for all } m \in \mathbb{N}.$$

2. Let $P(x)$ be a monic polynomial of degree n with nonnegative coefficients and the free term equal to 1. Prove that if all the roots of $P(x)$ are real, then $P(x) \geq (x+1)^n$ holds for every $x \geq 0$.

3. Show that there is a unique positive integer which consists of the digits 2 and 5, having 2005 digits and divisible by 2^{2005}.

4. Let P and Q be points on the sides BC and CD of a convex quadrilateral $ABCD$, respectively, such that $\angle BAP = \angle DAQ$. Prove that the triangles ABP and ADQ have equal area if and only if the line joining their orthocenters is perpendicular to AC.

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