12-th Croatian National Mathematical Competition 2003

High School

Pula, May 7–10, 2003

1-st Grade

1. Show that a triangle whose side lengths are prime numbers cannot have an integer area.

2. Show that if x,y,z are positive numbers with product 1 and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \geq x+y+z$, then
$$\frac{1}{x^k}+\frac{1}{y^k}+\frac{1}{z^k} \geq x^k+y^k+z^k \quad \text{for all } k \in \mathbb{N}.$$

3. In an isosceles triangle with base a, lateral side b and height to the base v, it holds that $\frac{a}{2}+v \geq b\sqrt{2}$. Find the angles of the triangle. Compute its area if $b = 8\sqrt{2}$.

4. How many divisors of 30^{2003} are there which do not divide 20^{2000}?

2-nd Grade

1. Find all pairs of real numbers (x,y) satisfying
$$(2x+1)^2+y^2+(y-2x)^2 = \frac{1}{3}\,.$$

2. Let M be a point inside a square $ABCD$ and A_1,B_1,C_1,D_1 be the second intersection points of AM,BM,CM,DM with the circumcircle of the square. Prove that $A_1B_1 \cdot C_1D_1 = A_1D_1 \cdot B_1C_1$.

3. For positive numbers $a_1,a_2,\dots,a_n$ ($n \geq 2$) denote $s = a_1+\dots+a_n$. Prove that
$$\frac{a_1}{s-a_1}+\dots+\frac{a_n}{s-a_n} \geq \frac{n}{n-1}\,.$$

4. Find the least possible cardinality of a set A of natural numbers, the smallest and greatest of which are 1 and 100, and having the property that every element of A except for 1 equals the sum of two elements of A.

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3-rd Grade

1. Let a, b, c be the sides of triangle ABC and α, β, γ be the corresponding angles.
   (a) If $\alpha = 3\beta$, prove that $(a^2 - b^2)(a - b) = bc^2$.
   (b) Is the converse true?
2. For every integer $n > 2$, prove the equality $\left[\dfrac{n(n+1)}{4n-2}\right] = \left[\dfrac{n+1}{4}\right]$.
3. In a tetrahedron $ABCD$, all angles at vertex D are equal to α and all dihedral angles between faces having D as a vertex are equal to φ. Prove that there exists a unique α for which $\varphi = 2\alpha$.
4. Given 8 unit cubes, 24 of their faces are painted in blue and the remaining 24 faces in red. Show that it is always possible to assemble these cubes into a cube of edge 2 on whose surface there are equally many blue and red unit squares.

4-th Grade

1. Let I be a point on the bisector of angle BAC of a triangle ABC. Points M, N are taken on the respective sides AB and AC so that $\angle ABI = \angle NIC$ and $\angle ACI = \angle MIB$. Show that I is the incenter of triangle ABC if and only if points M, N and I are collinear.
2. A sequence $(a_n)_{n\ge 0}$ satisfies $a_{m+n} + a_{m-n} = \frac{1}{2}(a_{2m} + a_{2n})$ for all integers m, n with $m \ge n \ge 0$. Given that $a_1 = 1$, find a_{2003}.
3. The natural numbers 1 through 2003 are arranged in a sequence. We repeatedly perform the following operation: If the first number in the sequence is k, the order of the first k terms is reversed. Prove that after several operations number 1 will occur on the first place.
4. Prove that the number $\dbinom{n}{p} - \left[\dfrac{n}{p}\right]$ is divisible by p for every prime number p and integer $n \ge p$.

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