11-th Croatian National Mathematical Competition 2002

High School

Zadar, May 2–5, 2002

1-st Grade

1. The length of the middle line of a trapezoid is 4 and the angles at one of the bases are 40° and 50°. Determine the lengths of the bases if the distance between their midpoints is 1.

2. Prove that for any positive numbers a,b,c and any nonnegative integer p
$$a^{p+2}+b^{p+2}+c^{p+2}\geq a^pbc+b^pca+c^pab.$$

3. Find all triples (x,y,z) of natural numbers that verify the equation
$$2x^2y^2+2y^2z^2+2z^2x^2-x^4-y^4-z^4=576.$$

4. A disc is divided into 30 segments which are labelled by $50,100,150,\dots,1500$ in some order. Show that there always exist three successive segments, the sum of whose labels is at least 2350.

2-nd Grade

1. Solve the equation
$$(x^2+3x-4)^3+(2x^2-5x+3)^3=(3x^2-2x-1)^3.$$

2. Let a,b,c be real numbers greater than 1. Prove the inequality
$$\log_a\left(\frac{b^2}{ac}-b+ac\right)\log_b\left(\frac{c^2}{ab}-c+ab\right)\log_c\left(\frac{a^2}{bc}-a+bc\right)\geq 1.$$

3. If two triangles with side lengths a,b,c and a',b',c' and the corresponding angles α,β,γ and α',β',γ' satisfy $\alpha+\alpha'=\pi$ and $\beta=\beta'$, prove that $aa'=bb'+cc'$.

4. Find all natural numbers n for which the equation $\frac{1}{x}+\frac{1}{y}=\frac{1}{n}$ has exactly five solutions (x,y) in the set of natural numbers.

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3-rd Grade

1. In triangle ABC, the angles $\alpha = \angle A$ and $\beta = \angle B$ are acute. The isosceles triangle ACD and BCD with the bases AC and BC and $\angle ADC = \beta$, $\angle BEC = \alpha$ are constructed in the exterior of the triangle ABC. Let O be the circumcenter of $\triangle ABC$. Prove that $DO + EO$ equals the perimeter of triangle ABC if and only if $\angle ACB$ is right.
2. Prove that a natural number can be written as a sum of two or more consecutive positive integers if and only if that number is not a power of two.
3. Points E and F are taken on the diagonals AB_1 and CA_1 of the lateral faces ABB_1A_1 and CAA_1C_1 of a triangular prisma $ABCA_1B_1C_1$ so that $EF \parallel BC_1$. Find the ratio of the lengths of EF and BC_1.
4. Among the n inhabitants of an island, every two are either friends or enemies. Some day, the chief of the island orders that each inhabitant (including himself) makes and wears a necklace consisting of marbles, in such a way that the necklaces of two friends have at least one marble of the same type and that the necklaces of two enemies differ at all marbles. (A necklace may also be marbleless). Show that the chief's order can be achieved by using $[n^2/4]$ different types of stones, but not necessarily by using fewer types.

4-th Grade

1. For each x with $|x| < 1$, compute the sum of the series
$$1 + 4x + 9x^2 + \cdots + n^2x^{n-1} + \cdots$$
2. Consider the cube with the vertices $A(1,1,1), B(-1,1,1), C(-1,-1,1), D(1,-1,1)$ and A', B', C', D' symmetric to A, B, C, D respectively with respect to the origin O. Let T be a point not on the circumsphere of the cube and let $OT = d$. Denote $\alpha = \angle ATA'$, $\beta = \angle BTB'$, $\gamma = \angle CTC'$, $\delta = \angle DTD'$. Prove that
$$\tan^2\alpha + \tan^2\beta + \tan^2\gamma + \tan^2\delta = \frac{32d^2}{(d^2-3)^2}.$$
3. Let $f(x) = x^{2002} - x^{2001} + 1$. Prove that for every positive integer m, the numbers $m, f(m), f(f(m)), \ldots$ are pairwise coprime.
4. Let $(a_n)_{n\in\mathbb{N}}$ be an increasing sequence of positive integers. A term a_k in the sequence is said to be *good* if it is a sum of some other terms (not necessarily distinct). Prove that all terms of the sequence, apart from finitely many of them, are good.

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