3-rd Czech–Slovak Match 1997

Bilovec, June 16–19, 1997

1. Points K and L are chosen on the sides AB and AC of an equilateral triangle ABC such that $BK = AL$. Segments BL and CK intersect at P. Determine the ratio $AK : KB$ for which the segments AP and CK are perpendicular.

2. In a community of more than six people each member exchanges letters with exactly three other members of the community. Show that the community can be partitioned into two nonempty groups so that each member exchanges letters with at least two members of the group he belongs to.

3. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(f(x)+y) = f(x^2 - y) + 4f(x)y \quad \text{for all } x, y \in \mathbb{R}.$$

4. Is it possible to place 100 balls in space so that no two of them have a common interior point and each of them touches at least one third of the others?

5. The sum of several integers (not necessarily distinct) equals 1492. Decide whether the sum of their seventh powers can equal (a) 1996; (b) 1998.

6. In a certain language there are only two letters, A and B. The words of this language obey the following rules:

   (i) The only word of length 1 is A;

   (ii) A sequence of letters $X_1X_2\dots X_{n+1}$, where $X_i \in \{A, B\}$ for each i, forms a word of length $n+1$ if and only if it contains at least one letter A and is not of the form WA for a word W of length n.

   Show that the number of words consisting of 1998 A's and 1998 B's and not beginning with AA equals $\binom{3995}{1997} - 1$.

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