1-st Czech–Polish–Slovak Match 2001

Bílovec, June 14–15, 2001

1. Prove that for any positive numbers $a_1, a_2, \dots, a_n$ $(n \geq 2)$

$$(a_1^3+1)(a_2^3+1)\cdots(a_n^3+1) \geq (a_1^2a_2+1)(a_2^2a_3+1)\cdots(a_n^2a_1+1).$$

2. A triangle ABC has acute angles at A and B. Isosceles triangles ACD and BCE with bases AC and BC are constructed externally to triangle ABC such that $\angle ADC = \angle ABC$ and $\angle BEC = \angle BAC$. Let S be the circumcenter of $\triangle ABC$. Prove that the length of the polygonal line DSE equals the perimeter of triangle ABC if and only if $\angle ACB$ is right.

3. Let n and k be positive integers such that $n/2 < k \leq 2n/3$. Find the least number m for which it is possible to place m pawns on m squares of an $n \times n$ chessboard so that no column or row contains a block of k adjacent unoccupied squares.

4. Distinct points A and B are given on the plane. Consider all triangles ABC in this plane on whose sides BC, CA points D, E respectively can be taken so that

   (i) $\frac{BD}{BC} = \frac{CE}{CA} = \frac{1}{3}$;

   (ii) points A, B, D, E lie on a circle in this order.

   Find the locus of the intersection points of lines AD and BE.

5. Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy

$$f(x^2+y) + f(f(x)-y) = 2f(f(x)) + 2y^2 \quad \text{for all } x, y \in \mathbb{R}.$$

6. Points with integer coordinates in cartesian space are called *lattice* points. We color 2000 lattice points blue and 2000 other lattice points red in such a way that no two blue-red segments have a common interior point (a segment is *blue-red* if its two endpoints are colored blue and red). Consider the smallest rectangular parallelepiped that covers all the colored points.

   (a) Prove that this rectangular parallelepiped covers at least 500,000 lattice points.

   (b) Give an example of a coloring for which the considered rectangular parallelepiped covers at most 8,000,000 lattice points.