Chinese IMO Team Selection Test 1994

Time: 4.5 hours each day.

First Day

1. Determine all sets of four natural numbers such that the product of any three of them divided by the remaining number gives the remainder 1.

2. An $n \times n$ array of numbers is called an *n-code* if the numbers in every row and every column form an arithmetic progression. A set of squares in the array is called a *key* if, given the numbers in these squares, one can uniquely determine all numbers in the array.

   (a) Given $n \geq 4$, find the smallest natural s such that any s squares in an n-code form a key.

   (b) Given $n \geq 4$, find the smallest t such that any t squares on the diagonals of an n-code form a key.

3. Find the smallest natural number n with the following property: Whenever five vertices of a regular n-gon are colored red, there exists a line l such that the reflection of a red point in l is never a red point.

Second Day

4. Let be given $5n$ real numbers $r_i, s_i, t_i, u_i, v_i \geq 1$ $(1 \leq i \leq n)$, and let

$$R = \frac{1}{n}\sum_{i=1}^{n} r_i, \ \ S = \frac{1}{n}\sum_{i=1}^{n} s_i, \ \ T = \frac{1}{n}\sum_{i=1}^{n} t_i, \ \ U = \frac{1}{n}\sum_{i=1}^{n} u_i, \ \ V = \frac{1}{n}\sum_{i=1}^{n} v_i.$$

   Prove that

$$\prod_{i=1}^{n} \frac{r_i s_i t_i u_i v_i + 1}{r_i s_i t_i u_i v_i - 1} \geq \left(\frac{RSTUV + 1}{RSTUV - 1} \right)^n.$$

5. Given distinct prime numbers p and q and a natural number $n \geq 3$, find all integers a for which the polynomial $f(x) = x^n + ax^{n-1} + pq$ can be factored into two non-constant polynomials with integer coefficients.

6. A polygon S is called a *subpolygon* of T if all the vertices of S are in vertices of T.

   (a) Show that for an odd number $n \geq 5$ there exists m subpolygons of a convex n-gon (for some m) such that the subpolygons have no common edges, and every edge or diagonal of the n-gon is an edge of at least one of the subpolygons.

   (b) Find the smallest possible value of m.

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