Chinese IMO Team Selection Tests 1990

First Test

1. In a group of people, every m people have exactly one friend in common, where m is given. Friendship is a symmetric and non-reflexive relation. Find the number of friends of the person having the largest number of friends.

2. Finitely many polygons are placed in the coordinate plane. We say that the polygons are *properly placed* if for any two polygons there is a line through the origin cutting both of them. Find the smallest positive integer m for which it is always possible to draw m lines through the origin such that each polygon is cut at least once.

3. An operation $\circ$ on set S (with $a \circ b \in S$ for all $a, b \in S$) is such that:

   (i) $(a \circ b) \circ c = a \circ (b \circ c)$ for all $a, b, c \in S$;

   (ii) $a \circ b \neq b \circ a$ whenever $a \neq b$.

   (a) Prove that $(a \circ b) \circ c = a \circ c$ for all $a, b, c \in S$.

   (b) Give an example of operation $\circ$ on the set $S = \{1, 2, \dots, 1990\}$.

4. Number a has the property that for any real numbers x_1, x_2, x_3, x_4 there exist integers k_1, k_2, k_3, k_4 such that
$$\sum_{1 \le i < j \le 4} ((x_i - k_i) - (x_j - k_j))^2 \le a.$$
   Find the smallest a.

Second Test

1. If ABC is a triangle with $\angle C \ge 60^\circ$, prove that
$$(a+b)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \ge 4 + \frac{1}{\sin\frac{C}{2}}.$$

2. Find all functions $f, g, h : \mathbb{R} \mapsto \mathbb{R}$ satisfying $f(x) - g(y) = (x-y)h(x+y)$ for all $x, y \in \mathbb{R}$.

3. Prove that every integer power of 2 has a multiple whose all decimal digits are nonzero.

4. Find the maximum possible number of different circles that pass through four of the given seven points in the plane.

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