Chinese IMO Team Selection Tests 1988

First Test – May 3

1. What necessary and sufficient conditions must real numbers A,B,C satisfy in order that
$$A(x-y)(x-z)+B(y-z)(y-x)+C(z-x)(z-y)\geq 0$$
for all real numbers x,y,z?

2. Determine all functions $f:\mathbb{Q}\to\mathbb{C}$ such that
   (i) $f(x_1+x_2+\cdots+x_{1988})=f(x_1)f(x_2)\cdots f(x_{1988})$ for all rational numbers $x_1,x_2,\dots,x_{1988}$, and
   (ii) $\overline{f(1988)}f(x)=f(1988)\overline{f(x)}$ for all $x\in\mathbb{Q}$, where $\overline{z}$ denotes the complex conjugate of z.

3. In a triangle ABC with $\angle C=30^\circ$, D and E are points on AC and BC respectively such that $AD=BE=AB$. If O and I are the circumcenter and incenter of $\triangle ABC$, prove that $OI=DE$ and $OI\perp DE$.

4. Let k be a positive integer. Consider the set $S_k=\{(a,b)\mid a,b=1,2,\dots,k\}$. Two elements (a,b) and (c,d) of S_k are said to be indistinguishable if $a-c\equiv -1,0$ or $1 \pmod k$ and $b-d\equiv -1,0$ or $1 \pmod k$. Let r_k be the greatest possible number of pairwise distinguishable elements of S_k.
   (a) Find r_5 with proof.
   (b) Find r_7 with proof.
   (c) Find r_k in general (no proof needed).

Second Test – May 4

1. Define $x_n=3x_{n-1}+2$ for all positive integers n. Prove that an integer value can be chosen for x_0 so that x_{100} is divisible by 1988.

2. Let $ABCD$ be a fixed trapezoid with $AB\parallel CD$ and let M,N be fixed points on side AB with M between A and N. For a variable point P on side CD, ND meets AP and MC at E and F and BP meets MC at G, respectively. For which P is the area of quadrilateral $PEFG$ maximal?

3. A polygon in the $xy-$plane has area greater than n. Prove that it contains some points (x_i,y_i), $i=1,2,\dots,n+1$, such that x_i-x_j and y_i-y_j are integers for all i,j.

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4. With u, v as input, a machine generates $uv + v$ as output. In the first operation, the only operations that can be used are $-1, 1$ and a fixed real number c. In later operations, numbers generated in preceeding operations can also be used. Prove that for any polynomial $f(x) = a_0x^n + \cdots + a_n$ with integer coefficients the machine can generate $f(c)$ as output after finitely many operations.

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