Chinese IMO Team Selection Tests 1987

First Test

1. For all positive integers k find the smallest positive integer $f(k)$ for which there exist 5 sets $S_1, S_2, \ldots, S_5$ satisfying:

   (i) $|S_i| = k$ for $i = 1, \ldots, 5$;

   (ii) $S_i \cap S_{i+1} = \emptyset$ for $i = 1, \ldots, 5$ (where $S_6 = S_1$);

   (iii) $|\bigcup_{i=1}^{5} S_i| = f(k)$.

   Generalize to $n \geq 3$ sets instead of 5.

2. A rectangular polygon $\mathscr{P}$ with 100 sides has the following properties: (i) all its sides are parallel to the coordinate axes, and (ii) all its sides have odd integral lengths. Prove that the area of $\mathscr{P}$ is odd.

3. Define the sequence (r_n) by $r_1 = 1$ and $r_n = r_1 r_2 \cdots r_{n-1}$ for $n \geq 2$. Prove that if $a_1, a_2, \ldots, a_n$ are positive integers such that $\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} < 1$, then

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} < \frac{1}{r_1} + \frac{1}{r_2} + \cdots + \frac{1}{r_n}.$$

Second Test

1. Given a convex figure $\mathscr{S}$ in the Cartesian plane that is symmetric with respect to both axis, let $\mathscr{A}$ be a rectangle of the maximum possible area lying entirely within $\mathscr{S}$. Let λ be the smallest ratio of the similitude with respect to the center of $\mathscr{A}$ such that the image of $\mathscr{A}$ under this similitude covers $\mathscr{S}$. Find the largest value of λ over all figures $\mathscr{S}$.

2. Find the positive integers n for which the equation $x^3 + y^3 + z^3 = nx^2y^2z^2$ has positive integer solutions.

3. Prove that in every simple graph with $2n$ vertices and $n^2 + 1$ edges ($n \geq 3$) there exist four vertices which are connected with each other by at least five edges.

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