Chinese IMO Team Selection Tests 1986

First Test

1. Let $ABCD$ be a cyclic convex quadrilateral and let I_A, I_B, I_C, I_D be the incenters of the triangles BCD, ACD, ABD, ABC, respectively. Show that $I_AI_BI_CI_D$ is a rectangle.

2. Let a_i, b_i $(i = 1, 2, \dots, n)$ be real numbers. Prove that the following two statements are equivalent:

   (i) $\sum_{k=1}^n a_k x_k \le \sum_{k=1}^n b_k x_k$ whenever $x_1 \le x_2 \le \cdots \le x_n$;

   (ii) $\sum_{k=1}^K a_k \le \sum_{k=1}^K b_k$ for $K = 1, 2, \dots, n-1$ and $\sum_{k=1}^n a_k = \sum_{k=1}^n b_k$.

3. For a natural number $A = \overline{a_n a_{n-1} \dots a_0}$ in decimal expansion, denote $f(A) = 2^n a_0 + 2^{n-1} a_1 + \cdots + a_n$. Define $A_1 = f(A)$ and $A_{i+1} = f(A_i)$ for $i \in \mathbb{N}$. Show that:

   (a) There exists a positive integer k for which $A_{k+1} = A_k$.

   (b) Determine the value of this A_k if $A = 19^{86}$.

4. A triangle ABC has a right angle at C. Show that, given any n points inside the triangle, we can denote them by $P_1, P_2, \dots, P_n$ such that

$$P_1P_2^2 + P_2P_3^2 + \cdots + P_{n-1}P_n^2 \le AB^2.$$

Second Test

1. Points P and Q on sides AB, AD respectively of a unit square are taken so that the perimeter of triangle APQ is 2. Find the angle PCQ.

2. Points E, F, G are chosen on the respective edges AB, AC and AD of a tetrahedron $ABCD$. Let S_{XYZ} denote the area and P_{XYZ} the perimeter of triangle XYZ. Prove that:

   (a) $S_{EFG} \le \max S_{ABC}, S_{ABD}, S_{ACD}, S_{BCD}$;

   (b) $S_{EFG} \le \max P_{ABC}, P_{ABD}, P_{ACD}, P_{BCD}$.

3. For $n \ge 3$ real numbers $x_1, x_2, \dots, x_n$, denote $p = \sum_{i=1}^n x_i$ and $q = \sum_{1 \le i < j \le n} x_i x_j$. Prove that:

   (a) $\dfrac{n-1}{n} p^2 - 2q \ge 0$;

   (b) $\left| x_i - \dfrac{p}{n} \right| \le \dfrac{n-1}{n} \sqrt{p^2 - \dfrac{2n}{n-1} q}$ for each $i = 1, \dots, n$.

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4. On a circle are marked $4k$ distinct points, arbitrarily labelled from 1 to $4k$.

   (a) Show that one can draw $2k$ pairwise disjoint chords with endpoints at the marked points so that the labels of the endpoints of each chord differ by at most $3k-1$.

   (b) Show that the bound $3k-1$ cannot be improved.

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