Chinese IMO Team Selection Test 2000

Time: 4.5 hours each day.

First Day

1. Let ABC be a triangle with $AB = AC$. Let D,E be points on AB,AC respectively such that $DE = AC$. Line DE meets the circumcircle of triangle ABC at point T. Let P be a point on AT. Prove that $PD + PE = AT$ if and only if P lies on the circumcircle of triangle ADE.

2. Given positive integers k,m,n with $1 \le k \le m \le n$, evaluate
$$\sum_{i=0}^{n} \frac{1}{n+k+i} \cdot \frac{(m+n+i)!}{i!(n-i)!(m+i)!}.$$

3. For an integer $a \ge 2$, let N_a denote the number of positive integers k with the following property: the sum of squares of digits of k in base a representation equals k. Prove that:
   (a) N_a is odd;
   (b) For every $M \in \mathbb{N}$, there exists an integer $a \ge 2$ such that $N_a \ge M$.

Second Day

4. Let F be the set of all polynomials $\Gamma(x)$ with integer coefficients such that the equation $\Gamma(x) = 1$ has integer roots. Given a positive integer k, find the smallest integer $m(k) > 1$ such that there exists $\Gamma \in F$ for which $\Gamma(x) = m(k)$ has exactly k distinct integer roots.

5. (a) Let a,b be real numbers. Define the sequences (x_n) and (y_n) by $x_0 = 1$, $y_0 = 0$ and $x_{k+1} = ax_k - by_l$, $y_{k+1} = x_k - ay_k$ for $k = 0,1,2,\dots$ Prove that
$$x_k = \sum_{l=0}^{[k/2]} (-1)^l a^{k-2l} (a^2+b)^l \lambda_{k,l},$$
   where $\lambda_{k,l} = \sum_{m=l}^{[k/2]} \binom{k}{2m} \binom{m}{l}$.
   (b) Let $u_k = \sum_{l=0}^{[k/2]} \lambda_{k,l}$. For a fixed positive integer m, denote the remainder of u_k divided by 2^m as $z_{m,k}$. Prove that the sequence $(z_{m,k})$, $k = 0,1,2,\dots$ is periodic, and find the smallest period.

6. Given a positive integer n, let us denote $M = \{(x,y) \in \mathbb{Z} \mid 1 \le x,y \le n\}$. Consider all functions $f : M \to \mathbb{Z}$ which satisfy:

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(i) $f(x,y) \geq 0$ for all x,y;

(ii) $\sum_{y=1}^{n} f(x,y) = n-1$ for each $x = 1,\dots,n$;

(iii) if $f(x_1,y_1)f(x_2,y_2) > 0$ then $(x_1-x_2)(y_1-y_2) \geq 0$.

Find the number $N(n)$ of functions with the above properties. Determine the value of $N(4)$.

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