30-th Canadian Mathematical Olympiad 1998

1. Determine the number of real solutions a to the equation
$$\left[\frac{1}{2}a\right]+\left[\frac{1}{3}a\right]+\left[\frac{1}{5}a\right]=a\,.$$

2. Find all real numbers x such that
$$x=\sqrt{x-\frac{1}{x}}+\sqrt{1-\frac{1}{x}}.$$

3. Let $n \geq 2$ be a natural number. Show that
$$\frac{1}{n+1}\left(1+\frac{1}{3}+\cdots+\frac{1}{2n-1}\right)>\frac{1}{n}\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2n}\right).$$

4. Let ABC be a triangle with $\angle BAC=40^\circ$ and $\angle ABC=60^\circ$. Let D and E be the points lying on the sides AC and AB, respectively, such that $\angle CBD=40^\circ$ and $\angle BCE=70^\circ$. Lines BD and CE intersect at point F. Show that AF is perpendicular to BC.

5. Let m be a positive integer. Define the sequence a_n by $a_0=0$, $a_1=m$ and $a_{n+1}=m^2a_n-a_{n-1}$ for each $n\in\mathbb{N}$. Prove that an ordered pair (a,b) of non-negative integers, with $a\leq b$, gives a solution to
$$\frac{a^2+b^2}{ab+1}=m^2$$
if and only if (a,b) is of the form (a_n,a_{n+1}) for some $n\geq 0$.

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