5th Canadian Mathematical Olympiad 1973

1. (a) Find all real x such that $x < 1/(4x)$ and $x < 0$.
   (b) What is the greatest integer x satisfying $4x + 13 < 0$ and $x^2 + 3x > 16$?
   (c) Give a rational number between $11/24$ and $6/13$.
   (d) Express 100000 as a product of two integers not divisible by 10.
   (e) Without using logarithm tables evaluate $\dfrac{1}{\log_2 36} + \dfrac{1}{\log_3 36}$.

2. Solve the equation $|x+3| - |x-1| = x+1$ in the set of real numbers.

3. Prove that if p and $p+2$ are prime numbers greater than 3 then $p+1$ is divisible by 6.

4. The diagonals $P_0P_3, P_1P_3, P_3P_6, P_4P_6, P_6P_0, P_7P_0$ divide a convex 9-gon $P_0P_1\ldots P_8$ into 7 triangles. In how many ways is it possible to label these triangles with $\triangle_1, \triangle_2, \ldots, \triangle_7$ so that P_i is a vertex of $\triangle_i$ for each $i = 1, \ldots, 7$? Justify your answer.

5. For every $n \in \mathbb{N}$ let us denote $h(n) = 1 + \dfrac{1}{2} + \cdots + \dfrac{1}{n}$. Prove that for all n,

$$n + h(1) + h(2) + \cdots + h(n-1) = nh(n).$$

6. Let A, B be fixed points on a circle not collinear with its center O, and XY be a variable diameter. Find the locus of the intersection P of lines AX and BY.

7. Observe that $\dfrac{1}{1} = \dfrac{1}{2} + \dfrac{1}{2}$, $\dfrac{1}{2} = \dfrac{1}{3} + \dfrac{1}{6}$, $\dfrac{1}{3} = \dfrac{1}{4} + \dfrac{1}{12}$ etc. State and prove a general law suggested by these examples. Also prove that for any integer $n > 1$ there exist $i, j \in \mathbb{N}$ such that

$$\frac{1}{n} = \frac{1}{i(i+1)} + \frac{1}{(i+1)(i+2)} + \cdots + \frac{1}{j(j+1)}.$$

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com