2nd Canadian Mathematical Olympiad 1970

1. Find all triples (x,y,z) of real numbers such that the sum of any of these numbers with the product of the other two is 2.

2. In a triangle ABC, the angle A is obtuse. If $BC = a$, $AC = b$ and the altitudes from A and B have lengths h and k respectively, show that $a+h \geq b+k$ and find the conditions for equality.

3. A set of balls is given. Each ball is colored red or blue, and not all are of the same color. Each ball weights either 1 or 2 pounds, and not all have the same weight. Prove that there are two balls of different weights and different colors.

4. (a) Find all integers with the first decimal digit 6 which become 25 times smaller if the first digit is deleted.

   (b) Show that there is no integer which becomes 35 times smaller when the first digit is deleted.

5. A quadrilateral has one vertex on each of the sides of a unit square. If a,b,c,d are the sides of the quadrilateral, prove that
$$2 \leq a^2+b^2+c^2+d^2 \leq 4.$$

6. Given three non-collinear points A,B,C, construct a circle with center C such that the tangents from A and B to the circle are parallel.

7. Prove that from any five integers one can always choose three whose sum is divisible by 3.

8. Consider all segments of length 4 whose endpoints lie on the lines $y = x$ and $y = 2x$ respectively. Find an equation of the locus of the midpoints of these segments.

9. Let $f(n)$ be the sum of first n terms of the sequence $0,1,1,2,2,3,3,4,4,\dots$.

   (a) Give a formula for $f(n)$.

   (b) Show that $f(s+t) - f(s-t) = st$ for any integers s,t with $s > t > 0$.

10. Suppose that $f(x) = x^n + a_1x^{n-1} + \cdots + a_n$ is a polynomial with integer coefficients for which there exist distinct integers a,b,c,d such that $f(a) = f(b) = f(c) = f(d) = 5$. Prove that there is no integer k such that $f(k) = 8$.

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