# 41-st Canadian Mathematical Olympiad 2009

March 25, 2009

1. Given an $m \times n$ grid with squares labeled by 0 and 1, we say that a square labeled by 1 is *stranded* if there is some square labelled by 0 to its left in the same row, and some square labelled by 0 above it in the same column. The following picture shows a $4 \times 5$ grid with no stranded squares labelled by 1.

| | | | | |
|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 1 |
| 0 | 0 | 1 | 1 | 0 |
| 1 | 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 | 0 |

   Find a closed formula for the number of $2 \times n$ grids with no stranded squares labelled by 1.

2. Two circles of different radii are cut out of cardboard. Each circle is subdivided into 200 equal sectors. On each circle 100 sectors are painted white and the other 100 are painted black. The smaller circle is then placed on top of the larger circle, in such a way that their centers coincide. Show that one can rotate the small circle so that the sectors on the two circles line up and at least 100 sectors on the small circle lie over sectors of the same color on the big circle.

3. Define
$$f(x,y,z) = \frac{(xy+yz+zx)(x+y+z)}{(x+y)(x+z)(y+z)}.$$
   Determine the set of real number $r$ for which there exists a triplet $(x,y,z)$ of positive real numbers satisfying $f(x,y,z) = r$.

4. Find all pairs $(a,b)$ of integers such that $3^a + 7^b$ is a perfect square.

5. A set of points is marked on the plane, with the property that any three marked points can be covered with a disk of radius 1. Prove that the set of all marked points can be covered with a disk of radius 1.