33-rd Canadian Mathematical Olympiad 2001

1. *Randy:* “Hi Rachel, that’s an interesting quadratic equation you have written down. What are its roots?”
Rachel: “The roots are two positive integers. One of the roots is my age, and the other root is the age of my younger brother, Jimmy.”
Randy: “That is very neat! Let me see if I can figure out how old you and Jimmy are. That shouldn’t be too difficult since all of your coefficients are integers. By the way, I notice that the sum of the three coefficients is a prime number.”
Rachel: “Interesting. Now figure out how old I am.”
Randy: “Instead, I will guess your age and substitute it for x in your quadratic equation ... darn, that gives me -55, and not 0.”
Rachel: “Oh, leave me alone!”

   (a) Prove that Jimmy is two years old.

   (b) Determine Rachel’s age.

2. There is a row of squares numbered -10 to 10. Each square is coloured either red or white, and the sum of the numbers on the red squares is n. Maureen starts with a token on the square labeled 0. She then tosses a fair coin ten times. Every time she flips heads (resp. tails), she moves the token one square to the right (resp. left). At the end of the ten flips, the probability that the token finishes on a red square is a rational number a/b. Given that $a+b=2001$, determine the largest possible value for n.

3. Let ABC be a triangle with $AC>AB$. The perpendicular bisector of BC and the angle bisector of $\angle A$ meet at P. Let X and Y be the feet of perpendiculars from P to lines AB and AC, respectively, and let Z be the intersection point of XY and BC. Determine the value of BZ/ZC.

4. Let n be a positive integer. Nancy is given a rectangular table with positive integer entries. She is permitted to make either of the following two moves:

   (i) select a row and multiply each entry in this row by n;

   (ii) select a column and subtract n from each entry in this column.

   Find all possible values of n with the following property: Given any rectangular table, it is possible for Nancy to perform a finite sequence of moves to create a table in which each entry is 0.

5. Let P_0, P_1, P_2 be three points on a unit circle, where $P_1P_2 = t < 2$. For each $i \geq 3$, define P_i to be the circumcenter of $\triangle P_{i-1}P_{i-2}P_{i-3}$.

   (a) Prove that the points $P_1, P_5, P_9, P_{13}, \dots$ are collinear.

   (b) Let $x = P_1P_{1001}$ and $y = P_{1001}P_{2001}$ Determine all values of t for which $\sqrt[500]{x/y}$ is an integer.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com