32-nd Canadian Mathematical Olympiad 2000

1. At noon, Anne, Beth and Carmen begin running laps around a circular track of length 300m, all starting from the same point on the track. Each jogger maintains a constant speed in one of the two possible directions for an indefinite period of time. Show that if Anne's speed is different from the other two speeds, then at some later time Anne will be at least $100m$ away (the distance being measured along the arc) from each of the other runners.

2. Given a permutation $a_1, a_2, \dots, a_{100}$ of $1901, 1902, \dots, 2000$, we form the sequence of partial sums
$$s_k = a_1 + a_2 + \cdots + a_k, \qquad k = 1, 2, \dots, 100.$$
How many of these permutations will have no terms of the sequence $s_1, \dots, s_{100}$ divisible by three?

3. Consider any sequence $A = (a_1, a_2, \dots, a_{2000})$ of integers from the interval $[-1000, 1000]$. Suppose that the sum of entries in A is 1. Show that the sum of entries in some nonempty subsequence of A is zero.

4. Let $ABCD$ be a convex quadrilateral with
$$\angle CBD = 2\angle ADB, \qquad \angle ABD = 2\angle CDB \quad \text{and} \quad AB = CB.$$
Prove that $AD = CD$.

5. Suppose that the real numbers $a_1, a_2, \dots, a_{100}$ satisfy
$$a_1 \ge a_2 \ge \cdots \ge a_{100} \ge 0;$$
$$a_1 + a_2 \le 100, \qquad a_3 + a_4 + \cdots + a_{100} \le 100.$$
Determine the maximum possible value of $a_1^2 + a_2^2 + \cdots + a_{100}^2$, and find all sequences $a_1, \dots, a_{100}$ which achieve this maximum.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com