14-th Baltic Way

Riga, Latvia – November 2, 2003

1. Find all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ which for all $x \in \mathbb{Q}^+$ fulfill
$$f\left(\frac{1}{x}\right) = f(x) \quad \text{and} \quad \left(1+\frac{1}{x}\right) f(x) = f(x+1).$$

2. Prove that any real solution of $x^3 + px + q = 0$, where p,q are real numbers, satisfies the inequality $4qx \le p^2$.

3. If x,y,z are positive numbers with $xyz = 1$, prove the inequality
$$(1+x)(1+y)(1+z) \ge 2\left(1+\sqrt[3]{\frac{y}{x}}+\sqrt[3]{\frac{z}{y}}+\sqrt[3]{\frac{x}{z}}\right).$$

4. Let a,b,c be positive real numbers. Prove that
$$\frac{2a}{a^2+bc}+\frac{2b}{b^2+ca}+\frac{2c}{c^2+ab} \le \frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}.$$

5. The sequence (a_n) is defined by $a_1 = \sqrt{2}$, $a_2 = 2$, and $a_{n+1} = a_n a_{n-1}^2$ for $n \ge 2$. Prove that for every $n \ge 1$
$$(1+a_1)(1+a_2)\cdots(1+a_n) < \left(2+\sqrt{2}\right) a_1 a_2 \cdots a_n.$$

6. Let $n \ge 2$ and $d \ge 1$ be integers with $d \mid n$, and let $x_1, x_2, \dots, x_n$ be real numbers such that $x_1 + x_2 + \cdots + x_n = 0$. Show that there are at least $\binom{n-1}{d-1}$ choices of d indices $1 \le i_1 < i_2 < \cdots < i_d \le n$ such that $x_{i_1} + x_{i_2} + \cdots + x_{i_d} \ge 0$.

7. A subset X of $\{1,2,3,\dots,10000\}$ has the following property: If a,b are distinct elements of X, then $ab \notin X$. What is the maximal number of elements in X?

8. There are 2003 pieces of candy on a table. Two players alternately make moves. A move consists of eating one candy or half of the candies on the table (the "lesser half" if there are an odd number of candies). At least one candy must be eaten at each move. The loser is the one who eats the last candy. Which player has a winning strategy?

9. It is known that n is a positive integer and $n \le 144$. Ten questions of type "Is n smaller than a?" are allowed. Answers are given with a delay: For $i = 1,\dots,9$, the i-th question is answered only after the $(i+1)$-th question is asked. The answer to the tenth question is given immediately. Find a strategy for identifying n.

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10. A *lattice point* in the plane is a point with integral coordinates. The *centroid* of four points (x_i, y_i), $i = 1,2,3,4$, is the point $(\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4})$. Let n be the largest natural number for which there are n distinct lattice points in the plane such that the centroid of any four of them is not a lattice point. Prove that $n = 12$.

11. Is it possible to select 1000 points in the plane so that at least 6000 pairwise distances between them are equal?

12. Points M and N are taken on the sides BC and CD respectively of a square $ABCD$ so that $\angle MAN = 45^\circ$. Prove that the circumcenter of $\triangle AMN$ lies on AC.

13. In a rectangle $ABCD$ with $BC = 2AB$, E be the midpoint of BC and P an arbitrary inner point of AD. Let F and G be the feet of perpendiculars drawn correspondingly from A to BP and from D to CP. Prove that the points E, F, P, and G are concyclic.

14. Equilateral triangles AMB, BNC, CKA are constructed in the exterior of a triangle ABC. The perpendiculars from the midpoints of MN, NK, KM to the respective lines CA, AB, BC are constructed. Prove that these three perpendiculars pass through a single point.

15. The diagonals of a cyclic convex quadrilateral $ABCD$ intersect at P. A circle through P touches the side CD at its midpoint M and intersects the segments BD and AC again in points Q and R respectively. Let S be the point on segment BD such that $BS = DQ$. The line through S parallel to AB intersects AC at T. Prove that $AT = RC$.

16. Find all pairs of positive integers (a,b) such that $a-b$ is a prime number and ab is a perfect square.

17. All the positive divisors of a positive integer n are stored into an increasing array. Mary is writing a program which decides for an arbitrarily chosen divisor $d > 1$ whether it is a prime. Let n have k divisors not greater than d. Mary claims that it suffices to check divisibility of d by the first $\lceil k/2 \rceil$ divisors of n: d is prime if and only if none of them but 1 divides d. Is Mary right?

18. Every integer is to be colored blue, green, red, or yellow. Can this be done in such a way that if a,b,c,d are not all 0 and have the same color, then $3a-2b \neq 2c-3d$?

19. Let a and b be positive integers. Show that if a^3+b^3 is the square of an integer, then $a+b$ is not a product of two different prime numbers.

20. Suppose that the sum of all positive divisors of a natural number n, n excluded, plus the number of these divisors is equal to n. Prove that $n = 2m^2$ for some integer m.