3-rd Baltic Way

Vilnius, Lithuania – November 7, 1992

1. Let p and q be two consecutive odd prime numbers. Prove that $p+q$ is a product of at least three positive integers greater than 1 (not necessarily different).

2. Denote by $d(n)$ the number of all positive divisors of a positive integer n. Prove that there are infinitely many n for which $d(n)$ divides n.

3. Find an infinite non-constant arithmetic progression of positive integers such that each term is neither a sum of two squares, nor a sum of two cubes.

4. Is it possible to draw a hexagon with vertices in integer lattice points so that the squares of the lengths of its sides are six consecutive positive integers?

5. Given that $a^2+b^2+(a+b)^2 = c^2+d^2+(c+d)^2$, prove that $a^4+b^4+(a+b)^4 = c^4+d^4+(c+d)^4$.

6. Prove that the product of the 99 numbers $\frac{k^3-1}{k^3+1}$ for $k = 2,3,\dots,100$ is greater than $\frac{2}{3}$.

7. Let $a = \sqrt[1992]{1992}$. Which number is greater:
$$a^{a^{a^{\cdot^{\cdot^{\cdot^{a}}}}}}\Big\}1992 \quad \text{or} \quad 1992?$$

8. Find all integer solutions of the equation $2^x(4-x) = 2x+4$.

9. Prove that every polynomial $f(x) = x^3+ax^2+bx+c$ with $b<0$ and $ab = 9c$ has three different real roots.

10. Find all fourth degree polynomials $p(x)$ satisfying the following conditions:
    - (i) $p(x) = p(-x)$ for all x;
    - (ii) $p(x) \geq 0$ for all x;
    - (iii) $p(0) = 1$,
    - (iv) $p(x)$ has exactly two local minimum points x_1, x_2 and $|x_1-x_2| = 2$.

11. Show that there is a unique function $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that
    - (i) If $0 < q < \frac{1}{2}$ then $f(q) = 1+f\left(\frac{q}{1-2q}\right)$;
    - (ii) If $1 < q \leq 2$ then $f(q) = 1+f(q-1)$;
    - (iii) $f(q)f\left(\frac{1}{q}\right) = 1$ for all $q \in \mathbb{Q}^+$.

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12. Assume that $\varphi : \mathbb{N} \to \mathbb{N}$ is a bijective function such that there exists a finite limit

$$\lim_{n\to\infty} \frac{\varphi(n)}{n} = L.$$

What are the possible values of L?

13. Prove that for any positive numbers $x_1, x_2, \dots, x_n$ and $y_1, y_2, \dots, y_n$

$$\sum_{i=1}^{n} \frac{1}{x_i y_i} \geq \frac{4n^2}{\sum_{i=1}^{n}(x_i+y_i)^2}.$$

14. The (finitely many) towns in a country are connected by one direction roads. It is known that, for any two towns, one of them can be reached from the other one. Prove that there is a town from which all the remaining towns can be reached.

15. Noah has to fit eight species of animals into four cages of the ark. He plans to put species in each cage. For each of the species there are at most three other species with which it cannot safely share a cage. Prove that he can safely accomodate all the animals.

16. All faces of a convex polyhedron are parallelograms. Can the polyhedron have exactly 1992 faces?

17. A quadrilateral $ABCD$ is inscribed in a circle with radius 1 and AC as a diameter, and $BD = AB$. The diagonals AC and BD intersect at P. Given that $PC = \frac{2}{5}$, how long is the side CD?

18. Show that in a non-obtuse triangle the perimeter is always greater than two times the diameter of the circumcircle.

19. Two non-intersecting circles C_1 and C_2 in the plane touch a circle C internally at points A and B respectively. A common tangent t of C_1 and C_2 touches them at D and E respectively so that both C_1 and C_2 are on the same side of t. Let F be the intersection point of AD and BE. Show that F lies on C.

20. Let $a \leq b \leq c$ be the sides of a right triangle of perimeter $2p$. Show that the area of the triangle equals $S = p(p-c) = (p-a)(p-b)$.

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