2-nd Baltic Way

Tartu, Estonia – December 14, 1991

1. Find the smallest positive integer n having the property that for any n distinct integers $a_1, a_2, \dots, a_n$ the product of all differences $a_i - a_j$ $(i < j)$ is divisible by 1991.

2. Prove that $102^{1991} + 103^{1991}$ is not a proper power of an integer.

3. There are 20 cats priced from $12 to $15 and 20 sacks priced from 10 cents to $1 for sale, all of different prices. Prove that John and Peter can each buy a cat in a sack paying the same amount of money.

4. A polynomial p with integer coefficients is such that $p(-n) < p(n) < n$ for some integer n. Prove that $p(-n) < -n$.

5. For any positive numbers a, b, c prove the inequalities
$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq \frac{2}{a+b}+\frac{2}{b+c}+\frac{2}{c+a} \geq \frac{9}{a+b+c}.$$

6. Solve the equation $[x] \cdot \{x\} = 1991x$.

7. If α, β, γ are the angles of an acute-angled triangle, prove that
$$\sin\alpha + \sin\beta > \cos\alpha + \cos\beta + \cos\gamma.$$

8. Let a, b, c, d, e be distinct real numbers. Prove that the equation
$$\begin{aligned}&(x-a)(x-b)(x-c)(x-d)+(x-a)(x-b)(x-c)(x-e)\\ &+(x-a)(x-b)(x-d)(x-e)+(x-a)(x-c)(x-d)(x-e)\\ &+(x-b)(x-c)(x-d)(x-e) = 0\end{aligned}$$
has four distinct real solutions.

9. Find the number of real solutions of the equation $ae^x = x^3$, where a is a real parameter.

10. Express the value of $\sin 3^\circ$ in radicals.

11. The integers from 1 to 1000000 are divided into two groups consisting of numbers with odd or even sums of digits respectively. Which group contains more numbers?

12. The vertices of a convex 1991-gon are enumerated with integers from 1 to 1991. Each side and diagonal of the 1991-gon is colored either red or blue. Prove that, for an arbitrary renumeration of vertices, one can find integers k and l such that the segment connecting the vertices numbered k and l before the renumeration has the same color as the segment connecting the vertices numbered k and l after the renumeration.

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13. An equilateral triangle is divided into 25 equal equilateral triangles labelled by 1 through 25. Prove that one can find two triangles having a common side whose labels differ by more than 3.

14. A castle has a number of halls and n doors. Every door leads into another hall or outside. Every hall has at least two doors. A knight enters the castle. In any hall, he can choose any door for exit except the one he just used to enter that hall. Find a strategy allowing the knight to get outside after visiting no more than $2n$ halls (a hall is counted each time it is entered).

15. In each of the squares of a chessboard an arbitrary integer is written. A king starts to move on the board. Whenever the king moves to some square, the number in that square is increased by 1. Is it always possible to make the numbers on the chessboard:

    (a) all even;

    (b) all divisible by 3;

    (c) all equal?

16. Two circles C_1 and C_2 with radii r_1 and r_2 touch each other externally and both touch a line l. A circle C_3 with radius $r_3 < r_1, r_2$ is tangent to l and externally to C_1 and C_2. Prove that
$$\frac{1}{\sqrt{r_3}} = \frac{1}{\sqrt{r_1}} + \frac{1}{\sqrt{r_2}}.$$

17. Let the coordinate planes have the reflection property. A ray falls onto one of them. How does the final direction of the ray after reflecting from all three coordinate planes depend on its initial direction?

18. Is it possible to place two non-intersecting tetrahedra of volume $\frac{1}{2}$ into a sphere with radius 1?

19. Three circles in the plane, whose interiors have no common point, meet each other at three pairs of points: A_1 and A_2, B_1 and B_2, and C_1 and C_2, where points A_2, B_2, C_2 lie inside the triangle $A_1B_1C_1$. Prove that
$$A_1B_2 \cdot B_1C_2 \cdot C_1A_2 = A_1C_2 \cdot C_1B_2 \cdot B_1A_2.$$

20. Consider two points $A(x_1, y_1)$ and $B(x_2, y_2)$ on the graph of the function $y = \frac{1}{x}$ such that $0 < x_1 < x_2$ and $AB = 2 \cdot OA$, where $O = (0,0)$. Let C be the midpoint of the segment AB. Prove that the angle between the x-axis and the ray OA is equal to three times the angle between the x-axis and the ray OC.