48-th Bulgarian Mathematical Olympiad 1999

Fourth Round – May 18–19, 1999

First Day

1. The faces of a box with integer edge lengths are painted green. The box is partitioned into unit cubes. Find the dimensions of the box if the number of unit cubes with no green face is one third of the total number of cubes. *(S. Grozdev)*

2. Let $(a_n)_{n=1}^{\infty}$ be a sequence of integers satisfying

$$(n-1)a_{n+1} = (n+1)a_n - 2(n-1) \quad \text{for all } n \geq 1.$$

If $2000 \mid a_{1999}$, find the smallest $n \geq 2$ such that $2000 \mid a_n$. *(O. Mushkarov, N. Nikolov)*

3. The vertices of a triangle have integer coordinates and one of its sides is of length $\sqrt{n}$, where n is a square-free natural number. Prove that the ratio of the circumradius and the inradius is an irrational number. *(O. Mushkarov, N. Nikolov)*

Second Day

4. Find the number of all integers n with $4 \leq n \leq 1023$ which contain no three consecutive equal digits in their binary representations. *(E. Kolev)*

5. The vertices A, B, C of an acute-angled triangle ABC lie on the sides B_1C_1, C_1A_1, A_1B_1 respectively of a triangle $A_1B_1C_1$ similar to the triangle ABC ($\angle A = \angle A_1$, etc.). Prove that the orthocenters of triangles ABC and $A_1B_1C_1$ are equidistant from the circumcenter of $\triangle ABC$. *(N. Nikolov)*

6. Prove that the equation $x^3 + y^3 + z^3 + t^3 = 1999$ has infinitely many integer solutions. *(G. Grigorov)*

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com