46-th Bulgarian Mathematical Olympiad 1997

Fourth Round – May 1997

First Day

1. For an integer $n \geq 2$ and $k = \left[\frac{n-2}{3}\right]$, consider the polynomial

$$P_n(x) = \binom{n}{2} + \binom{n}{5}x + \binom{n}{8}x^2 + \cdots + \binom{n}{3k+2}x^k.$$

   (a) Prove that $P_{n+3}(x) = 3P_{n+2}(x) - 3P_{n+1}(x) + (x+1)P_n(x)$.

   (b) Find all integers a such that $P_n(a^3)$ is divisible by $3^{\left[\frac{n-1}{2}\right]}$ for all $n \geq 2$.

2. Let M be the centroid of a triangle ABC. Prove the inequality

$$\sin\angle CAM + \sin\angle CBM \leq \frac{2}{\sqrt{3}}.$$

3. Let n and m be natural numbers and let $m + i = a_i b_i^2$ for $i = 1, 2, \ldots, n$, where a_i, b_i are natural numbers and a_i is not divisible by a square greater than 1. Find all n for which there exists an m such that $a_1 + a_2 + \cdots + a_n = 12$.

Second Day

4. If a, b, c are positive real numbers with $abc = 1$, prove the inequality

$$\frac{1}{1+a+b} + \frac{1}{1+b+c} + \frac{1}{1+c+a} \leq \frac{1}{2+a} + \frac{1}{2+b} + \frac{1}{2+c}.$$

5. In a triangle ABC, the bisectors of the angles at B and C meet the opposite sides at M and N respectively. The ray MN intersects the circumcircle of $\triangle ABC$ at D. Prove that

$$\frac{1}{BD} = \frac{1}{AD} + \frac{1}{CD}.$$

6. Let X be a set of $n+1$ elements, $n \geq 2$. Ordered n-tuples $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ formed from distinct elements of X are called *disjoint* if there exist distinct indices i, j such that $a_i = b_j$. Find the maximal number of pairwise disjoint n-tuples.

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