39-th Bulgarian Mathematical Olympiad 1990

Fourth Round

First Day

1. Consider the number obtained by writing the numbers $1, 2, \dots, 1990$ one after another. In this number every digit on an even position is omitted; in the so obtained number, every digit on an odd position is omitted; then in the new number every digit on an even position is omitted, and so on. What will be the last remaining digit?

2. Let be given a real number $\alpha \neq 0$. Show that there is a unique point P in the coordinate plane, such that for every line through P which intersects the parabola $y = \alpha x^2$ in two distinct points A and B, segments OA and OB are perpendicular (where O is the origin).

3. Let $n = p_1 p_2 \cdots p_s$, where $p_1, \dots, p_s$ are distinct odd prime numbers.

   (a) Prove that the expression
   $$F_n(x) = \prod \left(x^{\frac{n}{p_{i_1} \cdots p_{i_k}}} - 1 \right)^{(-1)^k},$$
   where the product goes over all subsets $\{p_{i_1}, \dots, p_{i_k}\}$ of $\{p_1, \dots, p_s\}$ (including itself and the empty set), can be written as a polynomial in x with integer coefficients.

   (b) Prove that if p is a prime divisor of $F_n(2)$, then either $p \mid n$ or $n \mid p - 1$.

Second Day

4. Suppose M is an infinite set of natural numbers such that, whenever the sum of two natural numbers is in M, one of these two numbers is in M as well. Prove that the elements of any finite set of natural numbers not belonging to M have a common divisor greater than 1.

5. Given a circular arc, find a triangle of the smallest possible area which covers the arc so that the endpoints of the arc lie on the same side of the triangle.

6. The base ABC of a tetrahedron $MABC$ is an equilateral triangle, and the lateral edges MA, MB, MC are sides of a triangle of the area S. If R is the circumradius and V the volume of the tetrahedron, prove that $RS \geq 2V$. When does equality hold?

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