38-th Bulgarian Mathematical Olympiad 1989

Fourth Round

First Day

1. In a triangle ABC, point O is the center of the excircle touching the side BC, while the other two excircles touch the sides AB and AC at points M and N respectively. A line through O perpendicular to MN intersects the line BC at P. Determine the ratio AB/AC, given that the ratio of the area of $\triangle ABC$ to the area of $\triangle MNP$ is $2R/r$, where R is the circumradius and r the inradius of $\triangle ABC$.

2. Prove that the sequence (a_n), where
$$a_n = \sum_{k=1}^{n} \left\{ \frac{[2^{k-\frac{1}{2}}]}{2} \right\} 2^{1-k},$$
converges, and determine its limit as $n \to \infty$.

3. Let p be a real number and $f(x) = x^p - x + p$. Prove that:
   (a) Every root α of $f(x)$ satisfies $|\alpha| < p^{\frac{1}{p-1}}$;
   (b) If p is a prime number, then $f(x)$ cannot be written as the product of two non-constant polynomials with integer coefficients.

Second Day

4. At each of the given n points on a circle, either +1 or -1 is written. The following operation is performed: between any two consecutive numbers on the circle their product is written, and the initial n numbers are deleted. Suppose that, for any initial arrangement of +1 and -1 on the circle, after finitely many operations all the numbers on the circle will be equal to +1. Prove that n is a power of two.

5. Prove that the perpendiculars, drawn from the midpoints of the edges of the base of a given tetrahedron to the opposite lateral edges, have a common point if and only if the circumcenter of the tetrahedron, the centroid of the base, and the top vertex of the tetrahedron are collinear.

6. Let x, y, z be pairwise coprime positive integers and $p \geq 5$ and q be prime numbers which satisfy the following conditions:
   (i) $6p$ does not divide $q-1$;
   (ii) q divides x^2+xy+y^2;
   (iii) q does not divide $x+y-z$.

   Prove that $x^p + y^p \neq z^p$.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com