37-th Bulgarian Mathematical Olympiad 1988

Fourth Round

First Day

1. Find all real parameters q for which there is a $p \in [0,1]$ such that the equation
$$x^4 + 2px^3 + (2p^2 - p)x^2 + (p-1)p^2x + q = 0$$
has four real roots.

2. Let n and k be natural numbers and p a prime number. Prove that if k is the exact exponent of p in $2^{2^n}+1$ (i.e. p^k divides $2^{2^n}+1$, but p^{k+1} does not), then k is also the exact exponent of p in $2^{p-1}-1$.

3. Let M be an arbitrary interior point of a tetrahedron $ABCD$, and let S_A, S_B, S_C, S_D be the areas of the faces BCD, ACD, ABD, ABC, respectively. Prove that
$$S_A \cdot MA + S_B \cdot MB + S_C \cdot MC + S_D \cdot MD \geq 9V,$$
where V is the volume of $ABCD$. When does equality hold?

Second Day

4. Let A, B, C be non-collinear points. For each point D of the ray AC, we denote by E and F the points of tangency of the incircle of $\triangle ABD$ with AB and AD, respectively. Prove that, as point D moves along the ray AC, the line EF passes through a fixed point.

5. The points of space are painted in two colors. Prove that there is a tetrahedron such that all its vertices and its centroid are of the same color.

6. Find all non-constant polynomials $p(x)$ satisfying $p(x^3+1) = p(x+1)^3$ for all x.

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