33-rd Bulgarian Mathematical Olympiad 1984

Fourth Round

First Day

1. Solve the equation $5^x \cdot 7^y + 4 = 3^z$ in nonnegative integers.

2. The diagonals of a trapezoid $ABCD$ with bases AB and CD intersect in a point O, and $AB/CD = k > 1$. The bisectors of the angles AOB, BOC, COD, DOA intersect AB, BC, CD, DA respectively at K, L, M, N. The lines KL and MN meet at P, and the lines KN and LM meet at Q. If the areas of $ABCD$ and OPQ are equal, find the value of k.

3. Points $P_1, P_2, \dots, P_n, Q$ are given in space ($n \geq 4$), no four of which are in a plane. Prove that if for any three distinct points $P_\alpha, P_\beta, P_\gamma$ there is a point P_δ such that the tetrahedron $P_\alpha P_\beta P_\gamma P_\delta$ contains the point Q, then n is an even number.

Second Day

4. Let $a, b, a_2, \dots, a_{n-2}$ be real numbers with $ab \neq 0$ such that all the roots of the equation
$$ax^n - ax^{n-1} + a_2 x^{n-2} + \cdots + a_{n-2} x^2 - n^2 bx + b = 0$$
are positive and real. Prove that these roots are all equal.

5. Let $0 < x_i < 1$ and $x_i + y_i = 1$ for $i = 1, 2, \dots, n$. Prove that
$$(1 - x_1 x_2 \cdots x_n)^m + (1 - y_1^m)(1 - y_2^m) \cdots (1 - y_n^m) > 1$$
for any natural numbers m and n.

6. Let be given a pyramid $SABCD$ whose base $ABCD$ is a parallelogram. Let N be the midpoint of BC. A plane λ intersects the lines SC, SA, AB at points P, Q, R respectively such that $\overrightarrow{CP}/\overrightarrow{CS} = \overrightarrow{SQ}/\overrightarrow{SA} = \overrightarrow{AR}/\overrightarrow{AB}$. A point M on the line SD is such that the line MN is parallel to λ. Show that the locus of points M, when λ takes all possible positions, is a segment of the length $\dfrac{\sqrt{5}}{2} SD$.

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