31-st Bulgarian Mathematical Olympiad 1982

Fourth Round

First Day

1. Find all pairs of natural numbers (n,k) such that $(n+1)^k - 1 = n!$.

2. Let n unit circles be given on a plane. Prove that on one of the circles there is an arc of length at least $2\pi/n$ not intersecting any other circle.

3. In a regular $2n$-gonal prism, bases $A_1A_2\dots A_{2n}$ and $B_1B_2\dots B_{2n}$ have circumradii equal to R. If the length of the lateral edge A_1B_1 varies, the angle between the line A_1B_{n+1} and the plane $A_1A_3B_{n+2}$ is maximal for $A_1B_1 = 2R\cos\frac{\pi}{2n}$.

Second Day

4. If $x_1, x_2, \dots, x_n$ are arbitrary numbers from the interval $[0,2]$, prove that
$$\sum_{i=1}^{n}\sum_{j=1}^{n} |x_i - x_j| \le n^2.$$
When is the equality attained?

5. Find all values of parameters a,b for which the polynomial
$$x^4 + (2a+1)x^3 + (a-1)^2x^2 + bx + 4$$
can be written as a product of two monic quadratic polynomials $\varphi(x)$ and $\psi(x)$, such that the equation $\psi(x) = 0$ has two distinct roots α,β which satisfy $\varphi(\alpha) = \beta$ and $\varphi(\beta) = \alpha$.

6. Find the locus of centroids of equilateral triangles whose vertices lie on sides of a given square $ABCD$.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com