30-th Bulgarian Mathematical Olympiad 1981

Fourth Round

First Day

1. Five points are given in space, no four of which are coplanar. Each of the segments connecting two of them is painted in white, green or red, so that all the colors are used and no three segments of the same color form a triangle. Prve that among these five points there is one at which segments of all the three colors meet.

2. Let ABC be a triangle such that the altitude CH and the sides CA, CB are respectively equal to a side and two distinct diagonals of a regular heptagon. Prove that $\angle ACB < 120^\circ$.

3. A quadrilateral pyramid is cut by a plane parallel to the base. Suppose that a sphere S is circumscribed and a sphere Σ inscribed in the obtained solid, and moreover that the line through the centers of these two spheres is perpendicular to the base of the pyramid. Show that the pyramid is regular.

Second Day

4. Let n be an odd positive integer. Prove that if the equation $\frac{1}{x}+\frac{1}{y}=\frac{4}{n}$ has a solution in positive integers x, y, then n has at least one divisor of the form $4k-1$, $k \in \mathbb{N}$.

5. Find all positive values of a, for which there is a number b such that the parabola $y = ax^2 - b$ intersects the unit circle at four distinct points. Also prove that for every such a there exists b such that the parabola $y = ax^2 - b$ intersects the unit circle at four distinct points whose x-coordinates form an arithmetic progression.

6. Planes $\alpha, \beta, \gamma, \delta$ are tangent to the circumsphere of a tetrahedron $ABCD$ at points A, B, C, D, respectively. Line p is the intersection of α and β, and line q is the intersection of γ and δ. Prove that if lines p and CD meet, then lines q and AB lie on a plane.

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