29-th Bulgarian Mathematical Olympiad 1980

Fourth Round

First Day

1. Show that there exists a unique sequence of decimal digits $p_0 = 5, p_1, p_2, \dots$ such that, for any k, the square of any positive integer ending with $\overline{p_k p_{k-1} \cdots p_0}$ ends with the same digits.

2. (a) Prove that the area of a given convex quadrilateral is at least twice the area of an arbitrary convex quadrilateral inscribed in it whose sides are parallel to the diagonals of the original one.

   (b) A tetrahedron with the total area S is intersected by a plane perpendicular to two opposite edges. If the area of the section is Q, prove that $S > 4Q$.

3. Each diagonal of the base and each lateral edge of a 9-gonal pyramid is colored either green or red. Show that there must exist a triangle with the vertices at vertices of the pyramid having all three sides of the same color.

Second Day

4. If a, b, c are arbitrary nonnegative real numbers, prove the inequality

$$a^3 + b^3 + c^3 + 6abc \geq \frac{1}{4}(a+b+c)^3$$

   with equality if and only if two of the numbers are equal and the third one equals zero.

5. Prove that the number of ways of choosing 6 among the first 49 positive integers, at least two of which are consecutive, is equal to $\binom{49}{6} - \binom{44}{6}$.

6. Show that if all lateral edges of a pentagonal pyramid are of equal length and all the angles between neighboring lateral faces are equal, then the pyramid is regular.

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